NATIONAL COMMITTEE ON PHYSICISTS AND MATHEMATICIANS

Recent action by the National Headquarters of the Selective Service System recognizes a critical shortage in the field of mathematics and sets up a new procedure whereby a national point of view may be brought to bear on the decisions of local boards in the cases of registrants who are occupied as mathematicians or are students of mathematics. This recent action of National Headquarters is best understood by reference to the historical background.

On November 7, 1942, National Headquarters announced a policy under which the Director of Selective Service might authorize the appointment of a National Committee in a critical scientific or specialized field, to assist the Selective Service System by reviewing affidavits for occupational classification; at the same time the National Committee on Physicists was established. On April 5, 1943, this policy was amended so as to apply to mathematicians as well as to physicists; *the National Committee on Physicists was replaced by the National Committee on Physicists and Mathematicians.* In addition to the two physicists who, with representatives of the National Roster of Scientific and Specialized Personnel and of the public, constituted the earlier Committee, the new National Committee on Physicists and Mathematicians will include two mathematicians.

This procedure involves no change in the fundamental principles underlying the granting of occupational deferment, which remains, in the first instance, a matter for the decision of each registrant's local board. The creation of the National Committee on Physicists and Mathematicians does, however, provide a means whereby the individual local boards will be advised concerning the national as well as the local needs in the critical fields of both physics and mathematics.

In Memorandum F (see Bulletin of the American Mathematical Society, March, 1943) attention was called to amendments made on December 14, 1942, to Occupational Bulletins No. 10 and No. 23. *The amended Bulletins remain in force and should be consulted in preparing requests for occupational classification of teachers or students of mathematics.*[1]

Mathematicians should note the changes in procedure made necessary by the establishment of the National Committee on Physicists

[1] The substance of the amended Bulletins No. 10 and No. 23 is to be found also in Activity and Occupation Bulletins No. 33-5 and No. 33-6, issued on March 1, 1943.

and Mathematicians. *The principal one is that the employer is now to file papers with the National Committee as well as with the employee's local board.* (See 3(b) below.) The following detailed outline, it is hoped, will be of value to mathematicians and their employers as the new procedure goes into effect.

1. Although full responsibility for presenting occupational deferment requests lies with employers,[2] the mathematician-registrant will be wise to see that his employer is fully informed of the new procedure established by Activity and Occupation Bulletin No. 35 as amended on April 5, 1943.

2. The employer must first decide whether he regards the registrant as a "necessary" man for whom he wishes to request occupational deferment (or continued occupational deferment). In the case of men with dependents, even if deferred at present, it is desirable to file information with the local boards establishing their occupational status as further ground for deferred classification.

3. For every registrant for whom occupational deferment (or continued deferment) is to be requested, the employer will:

(a) Fill out *completely* Selective Service Form 42 or 42A in triplicate. File a copy with the local board with a brief letter stating that the original has been sent to the National Committee on Physicists and Mathematicians as authorized in Selective Service Activity and Occupation Bulletin No. 35, issued March 1, 1943, and amended April 5, 1943.

(b) Send the original and one copy of Form 42 or 42A (see also (c) below) to the:

National Committee on Physicists and Mathematicians
National Roster of Scientific and Specialized Personnel
Washington, D. C.

(c) Send in addition to Form 42 or 42A, a "Report of Status" for the National Committee on Physicists and Mathematicians. Copies of these "Report of Status" blanks may be secured from the National Committee on Physicists and Mathematicians.

(d) For those *not now deferred*, this procedure should be followed at once. For those *now deferred*, this procedure should be followed *at least thirty days before the expiration of the deferment period.*

4. For every registrant for whom deferment is not to be requested

[2] Throughout this discussion the term "employer" means that person within an organization—industrial, educational, or governmental—who has been assigned the responsibility for handling deferment problems. In educational institutions, the head of the mathematics department must be an active adviser in matters involving both staff and students.

by his employer or college, a "Report of Status" in duplicate should be filled out and sent to the National Committee on Physicists and Mathematicians at once so that efforts may be made to place him where his talents and training will be of best service to the war effort.

The plan under which deferment applications for mathematicians are now to be made is announced in Activity and Occupation Bulletin No. 35, as amended on April 5, 1943. Extracts giving the essential features of this plan are published herewith:

"*Filing Affidavit—Occupational Classification.* When such a National Committee has been appointed for a scientific or specialized field, the employer or recognized university or college desiring occupational classification for a registrant who possesses the training, qualification, or skill in that field, or is in training or preparation therefor, may prepare Affidavit—Occupational Classification (Form 42 or 42A) in duplicate, file the copy with the local board in the usual manner, and forward the original to the National Committee. (Part I, C, 3)

"*Action by the National Committee.* When the National Committee receives a Form 42 or 42A, it will investigate the registrant. If, in its opinion, the registrant possesses the training, qualification, or skill and is a necessary man in an essential activity or a necessary man in training or preparation therefor, the National Committee is authorized to place a stamped endorsement prescribed by the Director of Selective Service on the original Form 42 or 42A and to file the form with the registrant's local board. (Part I, C, 4)

"*Consideration by local board.* When the original Form 42 or 42A, stamped by the National Committee, is received by the local board, it shall be considered as showing that the registrant has been investigated, that in the opinion of the National Committee he possesses the training, qualification, or skill required, and that, as the case may be, he is a necessary man in an essential activity or a necessary man in training or preparation in such scientific and specialized field. (Part I, C, 5)

"*Classification Advice* (*Form 59*). When a Form 42 or 42A, stamped by a National Committee, has been received for a registrant and the local board, nevertheless, classifies the registrant as available for military service or for assignment to work of national importance, the local board shall notify the National Committee, as well as the employer or university or college, by mailing to the National Committee and to the employer or university or college Classification Advice (Form 59). In such case the National Committee, as well as the employer or university or college, may appeal to the board of appeal from classification of the registrant. (Part I, C, 6)

"*National Committees.** The following National Committees for scientific and specialized fields presently are authorized by the Director of Selective Service:

PHYSICISTS AND MATHEMATICIANS

The National Committee on Physicists and Mathematicians, National Roster of Scientific and Specialized Personnel, Washington, D.C.

ENGINEERS AND CHEMISTS

The National Committee on Engineers and Chemists, National Roster of Scientific and Specialized Personnel, Washington, D. C."

(Part I, C, 7)

J. R. KLINE,
Secretary

May 11, 1943.

* As amended 4-5-43.

MATHEMATICAL QUESTIONS IN SEISMOLOGY

C. F. RICHTER

Seismologists, like mathematicians, are accustomed to dividing their subject into a pure and an applied branch. The desirability of such division is underlined by the manner in which non-seismologists frequently approach the subject. It is too often taken for granted that, because an earthquake is a motion of the ground, seismology must be chiefly concerned with the detailed analysis of that motion for its own sake. Such analysis is important in the engineering section of applied seismology; but it is unrepresentative of seismology as a whole, which deals largely with more remote problems: the causative mechanism of earthquakes, the distribution of their origins geographically and in depth, or the structure and physical condition of the interior of the earth.

Engineering applications are out of bounds for the present speaker. Briefly, they refer to the design of earthquake-resistant (not "earthquake-proof") structures. The problems differ from those of most structural engineering in being dynamical, not statical; buildings, bridges, and other structures must be designed with reference to their behavior when in violent vibratory motion. Further, horizontal forces must be considered more explicitly than is done in ordinary design, where the chief load forces, due to gravity, are vertical. Problems of principle are few; but the mechanical characteristics of large structures are only less complex than the motion of the ground in an earthquake. To arrive at solvable propositions it is usually necessary to idealize both sets of data to a degree which seriously limits the practical applicability. Consequently, progress has depended heavily on empirical investigation of the behavior of models on a shaking-table.

A very vocal offshoot of seismology is its application to geophysical prospecting—the investigation of subsurface structures by recording, with specially designed seismometers, the artificial earthquakes produced when explosives are detonated. Numerous books and periodicals deal with the problems of "geophysics"—which, in the vocabulary of the profession, means geophysical prospecting; a rather unwarranted restriction of the meaning of a necessary term, long in use with its proper significance as the physics of the earth. These problems present little of individual interest to the mathematician,

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since nearly all are specializations of problems and methods long familiar in connection with earthquakes. As an example, consider an idealization of "the reflection method." An explosion is set off at or near the surface of the earth, at a carefully recorded time. At unknown depth is a reflecting layer, of irregular form, which may be discontinuous in the mathematical sense. The arrival of elastic waves from the explosion, reflected at the named layer, is timed accurately at a number of points on the surface of the earth. Supposing the velocity known and constant, is the determination of arrival times at all points of a limited area (taken as plane) sufficient to determine the depth and form of the reflecting layer? More practically, what is the maximum information derivable from a limited number of observations? In practice, the explosions may be repeated at several different points.

An important division of seismology which as yet offers little hold to the mathematician is that dealing with the fundamental mechanisms of earthquakes—the possible physical origin of stresses giving rise to earthquakes, the mechanism of the actual process of fracturing. These problems belong to the field which has lately taken the name of "tectonophysics"; a section of the American Geophysical Union with that title was organized in 1939. This marks a movement to give greater precision, from the point of view of mathematical physics, to many of the concepts used in geology to describe the behavior of great masses of matter. The subject is being furthered by increasing experimental data on the physical properties of rocks and other material under great pressure. A detailed discussion would involve almost the whole of geophysics, of which seismology is only a small part. Reference may be made to the general manuals in the bibliography, and to the Transactions of the American Geophysical Union.

Theoretical seismology, like other branches of physics, is readily characterized mathematically by the occurrence of certain fundamental differential equations, the theory of which is involved in nearly every problem. These occur in three principal sets:

I. equations of motion of the recording instruments,

II. equations of motion of elastic solid bodies,

III. equations of propagation of elastic waves in the interior of the earth (usually assumed as spherical).

The first are the simplest. The motion of a seismometer is a particular case of the forced oscillations of vibrating systems, the theory of which is classical and well worked out. In its simplest form, the seismometer equation is a linear ordinary differential equation of the second order, with constant coefficients. The nonhomogeneous part

represents the disturbance due to an earthquake. The homogeneous term of zero order determines the free period of oscillation of the seismometer pendulum, and that of first order then gives the damping, or rate of decrease of the free oscillation. Further complication occurs when the seismometer displacement, instead of being recorded directly, operates an electromagnetic system which actuates a recording galvanometer. The free period and damping of the galvanometer then enter into the equations of motion of the coupled system, which in complete form are of fourth order. Since the coefficients are still constant, the problem is theoretically solved; but the practical application may become difficult and laborious if the damping coefficients are large enough to approach their critical values. Devices for clarifying the problem or shortening the computations are much to be desired.

Seismic waves are propagated through the whole earth; systematic investigation employs the data of many hundreds of earthquakes, as recorded at seismological stations in all parts of the formerly civilized world. The observations are affected by all ordinary causes of error, so that statistical treatment is necessary to ensure valid conclusions. Harold Jeffreys, at Cambridge, has devoted much of his distinguished ability for many years to the mathematical problems of geophysics and seismology; the professional mathematician will find no better introduction to these problems than his volume *The earth* [2]. Jeffreys' interest in the theory and technique of statistics as such has also appeared in many publications; in a recent volume [11] he has presented a detailed account of his somewhat unorthodox views. Seismologists have observed with much interest the application to their specialty of these highly refined techniques, which are for the most part beyond the range of their critical judgment. Jeffreys' specific results are mostly in excellent agreement with those of others who have not used his methods. He has sometimes claimed a surprisingly high degree of precision; but any difference of opinion on such points refers, not to the method of handling the data, but to the reliability of the observations as such. Others have attempted to reach the same objective by working so far as possible with original seismograms in preference to readings taken from them, and by selecting the data used for further conclusions through detailed consideration of each item, applying individual criteria not easily derived from a purely statistical treatment. Questions arise belonging not merely to statistics, but to the whole theory of scientific method. Reference may be made to Jeffreys' smaller volume [12], and to the excellent summary in Blake's paper [7].

Seismological data have at least one statistical idiosyncrasy; the errors are not symmetrical. The fundamental datum is the time of arrival of the first of a long succession of elastic waves at the recording station. If this first motion is small it may be missed, and a later time used instead. Early times, on the other hand, can arise only from the normally fluctuating causes of error, except occasionally when a seismologist, anxious not to overlook the true beginning, has measured a small extraneous disturbance preceding it.

The equations of set II are:

$$(1) \qquad \rho \frac{\partial^2 (u, v, w)}{\partial t^2} = (\lambda + \mu) \operatorname{grad} \Theta + \mu \nabla^2 (u, v, w),$$

$$(2) \qquad X_x = \lambda\Theta + 2\mu \frac{\partial u}{\partial x}, \quad Y_y = \lambda\Theta + 2\mu \frac{\partial v}{\partial y}, \quad Z_z = \lambda\Theta + 2\mu \frac{\partial w}{\partial z},$$

$$(3) \qquad X_y = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \quad Y_z = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right), \quad Z_x = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial \mu}{\partial z}\right).$$

u, v, w are the cartesian components of displacement of any elementary particle from its equilibrium position. ρ is the density; λ and μ are the Lamé constants of elasticity. Θ, the dilatation, is an abbreviation for the divergence of the vector (u, v, w). Equation (1), which stands for three, is thus a vector equation; it suffices for the theory of motion of a homogeneous isotropic elastic solid of infinite extent, without boundaries or discontinuities. The remaining equations give the components of the stress tensor in terms of the derivatives of the displacements. These are called for whenever it is necessary to apply boundary conditions.

Consider first the consequences of these classical equations. Generalizations and modifications will be cited later. The usual process of separating the displacement vector into two parts, one of which is irrotational, involves taking the divergence and the curl of equation (1); this yields the two wave equations:

$$(4) \qquad \frac{\partial^2 \Theta}{\partial t^2} = \frac{\lambda + 2\mu}{\rho} \nabla^2 \Theta,$$

$$(5) \qquad \frac{\partial^2 (\xi, \eta, \zeta)}{\partial t^2} = \frac{\mu}{\rho} \nabla^2 (\xi, \eta, \zeta).$$

Here $2(\xi,\eta,\zeta) = \operatorname{curl} (u,v,w)$.

This process proves the possible existence of two types of elastic waves. In the first, displacements are purely compressional or dilata-

tional, the displacement vector being in the direction of propagation of the wave. These are physically identical with sound waves. In the second type, displacement may be described as purely shearing, irrotational, or equivoluminal, being transverse to the direction of propagation and consequently exhibiting the phenomena of polarization as in optics. Whether these transverse waves should be called sound waves is a matter of terminology. Both longitudinal and transverse waves are observed in the earth. These are the only types of waves which can exist in an unlimited homogeneous isotropic elastic medium subject to equation (1). As soon as boundaries exist, however, other waves occur.

The simplest waves due to a boundary are the surface waves found by Lord Rayleigh [13]. Let the medium be semi-infinite with a free bounding surface at the plane $z=0$. Then a solution of (1) exists which contains a factor $\exp(az+ibx-ict)$ where a, b, c are real. This represents a wave propagated in the direction of x with velocity c/b; its amplitude decreases exponentially with increasing z if a is negative, so that on the positive side of the boundary the disturbance is practically confined to the vicinity of the free surface. The velocity c/b is determined by requiring the stress components (2) and (3) to vanish at the boundary $z=0$. This leads to a cubic equation in c^2/b^2. This equation is obviously characteristically connected with the partial differential equation (1); an exact analysis of this type of relationship might be of much interest.

Since the cubic equation is reached by squaring, some of its roots may be extraneous to the problem. One of the roots is always appropriate, and leads to the solution given by Rayleigh. The case when the other two roots are equal falls well within the admissible numerical range of the physical constants, so that with slightly different possible values these roots may be either both real or both complex. When both are complex, they represent a valid solution of the given equations and boundary conditions only if the amplitude becomes infinite for infinite z; this is not of physical interest. When both roots are real, a must be taken as imaginary; the solution is periodic and does not vanish at infinity. The velocities come out greater than those of the normal longitudinal and transverse waves represented by (4) and (5). This at first appears physically improbable. However, it readily appears that these solutions represent trains of plane longitudinal and transverse waves of the normal type and with the normal velocities, approaching the surface at specified angles; the high apparent velocity in the x direction is due to these angles of emergence. The result then appears trivial; it seems that the search for solutions satisfying

given conditions has led incidentally to a special case of another known solution. However, there is still a possible question: Does this mean physically that a disturbance below the plane free surface will actually produce, not merely the Rayleigh surface wave and the ordinary body waves with spherical wave front, but also these particular body waves with a different orientation? The same result was obtained by Somigliana [14] less directly, and has been applied by Caloi [15, 16] to explain certain observations of anomalously high velocities. To the present speaker it appears more probable that these observations have another explanation, and that the solutions in question correspond to no new physical phenomenon. The same case arises in the work of Jeans [26] to be cited presently.

The Rayleigh wave is only the simplest type of wave dependent on a discontinuity. Most seismograms show large waves of the type whose theory is due to A. E. H. Love [17]. These exist primarily in the space between two discontinuities; in the most important case the upper of these is the surface of the earth, while the lower is the base of the continental rocks. Similar waves may exist without a discontinuity, if the physical constants change rapidly with depth [18, 19]. This shows that a complete theory, now lacking, of the propagation of elastic waves in a nonhomogeneous but isotropic medium, would include several types which do not occur when the medium is homogeneous and unbounded.

More general boundary conditions have been applied. Stoneley and others [20–25] have worked on the theory of waves associated with an internal interface, and on the effect of multiple layering. Jeans [26] has considered the general problem of free oscillations of a spherically symmetrical earth. Surface waves of Rayleigh and Love types, as well as others, appear as particular solutions; the general solution is not worked through. Lamb, Sezawa, Nakano and others [27–41] have developed the theory of generation of bodily and surface waves from an initial disturbance of given form; this bears directly on the problem of earthquake causation.

Earthquake surface waves show considerable dispersion. The simple Rayleigh theory gives no dispersion, the velocity being fixed; however, the velocity depends on the frequency for Love waves [17, 18] and for Rayleigh waves on the surface of a heterogeneous medium [42–49]. The observed dispersion differs regionally, and is evidence for regional differences in crustal structure [4, chap. XII].

A quite different problem of much importance in the interpretation of seismograms is the following: Plane waves, either longitudinal or transverse, are incident at a given angle on the plane boundary be-

tween two media which differ in elastic constants and density. There are four derived plane waves; reflected longitudinal and transverse waves in the first medium, refracted longitudinal and transverse waves in the second medium. What is the ratio of the amplitude of each to that of the incident wave?

This problem was solved by Knott [50], and, a little more directly, by Zoeppritz [51]. The solution takes the form of four linear homogeneous algebraic equations in the amplitudes of the five waves, there being two cases according as the incident wave is longitudinal or transverse. Unfortunately, the coefficients are complicated, containing physical constants and trigonometric functions of the angles. Computation is laborious, and has been carried out only for a limited range of assumptions which approximate the actual conditions met with in seismology [52–54]. Our information is incomplete as to maxima and minima of the various amplitudes as functions of the physical parameters, and even as to the occurrence of zeros and singularities. It would appear that some of these questions might be dealt with directly from the equations of motion and the boundary conditions in differential form; but this has not been done.

On seismograms of earthquakes at distances between 100 and 1000 kilometers, the direct longitudinal wave, $\overline{P}$, is preceded by a longitudinal wave P_n, refracted through the material beneath the continental layers in which most earthquakes originate. At that depth it travels so fast that it arrives ahead of the direct wave. Analogous waves refracted horizontally through subsurface layers are observed in geophysical prospecting. Simplified forms of the theory usually indicate that such a horizontally refracted wave should exist as a type of surface wave in the lower medium, but that it should carry only a small part of the incident energy. While the direct wave is in fact larger, the observed refracted wave is usually not so small as the simple theory suggests. More precise discussion, such as that of Jeffreys [55, 56; 57, pp. 219–220] and Muskat [58] still does not satisfactorily represent the observations. Alternative explanations have been looked for. Recently O. von Schmidt [59, 60] has succeeded in producing essentially the same phenomenon in the laboratory. He emphasizes the analogy with the "head wave" in ballistics. A theory has been given by Joos and Teltow [61], but the subject is still incompletely investigated.

In practice, the equations of our set III are identical with the fundamental equations of geometrical optics. The dimensions of the earth and its larger structures are so vast compared with the wave lengths of bodily seismic waves (mostly of the order of ten kilometers) that

the observations can be discussed without much attention to phenomena of interference or diffraction. The chief difference from geometrical optics lies in the data, the more significant of which are determinations of time, whereas intensities and frequencies, though observed and used, are of only supplementary importance.

The customary general statement is Fermat's "principle of least time"

$$\delta \int ds/v = 0 \tag{6}$$

where the end points of the line integral are fixed during the variation. ds denotes the element of path, and v the velocity of propagation given as a function of position. The name is slightly misleading. Thus, when a shock originates within the earth and sends waves to a distant surface point, there are two principal paths of singly reflected waves satisfying the variation principle. Of these the more obvious and familiar, reflected nearly midway, corresponds to a maximum of the integral. The other, representing a minimum time, is reflected comparatively near the source; it is observed chiefly in earthquakes originating at great depth. Jeffreys [73, p. 560] has pointed out that in general seismic waves corresponding to maximum times are less sharp and well defined on the seismograms than those with minimum times.

Attention is called to the differential equation associated with the variation principle, and named by Bruns the equation of the "eikonal." For a spherically symmetrical earth it takes the form

$$(\partial t/\partial r)^2 + (1/r^2)(\partial t/\partial \theta)^2 = 1/v^2 \tag{7}$$

where r, θ, ϕ are polar coordinates with origin at the center and polar axis through the source of the earthquake. Since the velocity v is independent of ϕ, that coordinate drops out. t is the time of propagation, not along a fixed ray, but always along that ray which forms the path of least time from the source to the point (r, θ, ϕ).

This general equation can be made the basis for a complete discussion of the geometrical optics of seismology. More customarily, the wanted information is derived from a special consequence, namely Snell's law of refraction in its polar form, which states that along a given ray

$$(r \sin i)/v = \text{const.} = r_s/v_s. \tag{8}$$

Here i is the angle between the radius vector and the tangent to the ray. Except at the boundary of the core of the earth, velocity in-

creases with depth (since the elasticity increases more rapidly than the density, which alone would give the opposite effect). Seismic rays are convex toward the center of the earth, and each ray has a minimum radius vector r_s corresponding to $i=90°$; the velocity at this point on the ray is v_s.

The initial problem is to determine the velocity distribution within the earth. The observed times at seismological stations at varying angular distances from the source give t as a function of θ for $r=R$, the radius of the earth. The assumption of spherical symmetry is justified by the satisfactory agreement with all observations of tables giving t as a function of θ alone. The determination of $v(r)$ from $t(\theta)$ involves a Fredholm integral equation, of the type first solved by Abel [62, 63]. The method developed by Wiechert [64] and Herglotz [65], improved by Bateman [66], leads to integrals which are readily evaluated graphically or with a computing machine. The essential equations are:

$$\cosh q(x, \theta) = t'(x)/t'(\theta), \tag{9}$$

$$\log \frac{R}{r_s} = \frac{1}{\pi}\int_0^\theta q(x, \theta)dx, \tag{10}$$

$$r_s = v_s t'(\theta). \tag{11}$$

From the given $t(\theta)$ one proceeds to $t'(\theta)$ and thence to v_s as a function of r_s, identical with $v(r)$. General relations between the derivatives and singularities of $v(r)$ and $t(\theta)$ are of importance; few have been worked out.[1] A peculiar singularity is involved, which occurs in all similar problems. To state it simply, consider a source of disturbance at $x=0$, $y=0$, $z=-h$, the velocity v being a function of z only. Let the arrival time on the x-axis be given as $t(x)$. Then $t'(x)=0$ for $x=0$, so long as h is different from zero; but if $h=0$ then $t'(x)=1/v$ for $x=0$. Again, if h is not zero, $t''(x)=0$ for $x=0$; but it is not easily seen what the general behavior of the second derivative will be if $h=0$. This apparently artificial problem took on a physical significance in the following way [9, IV; 67, 68]: There is a well observed type of seismic wave which originates as a transverse wave, continues along a short path in the presumably liquid core of the earth as a longitudinal wave, and then emerges from the core as a transverse wave, in which form it reaches the recording stations. Shortening dis-

[1] The speaker gave the result by Gutenberg and Richter [9, II] for dv_s/dr_s in terms of t' and q. Subsequent discussion has made possible an adequate discussion of the higher derivatives, to be published in the near future. Correction to [9, II, p. 304]: $dv_s/dr_s=0$ for $I=\pi$, not for I infinite.

tance between earthquake source and observing station shortens the path in the core, until it finally vanishes and the wave in question ceases to exist. Let $t(\theta)$ now represent the times of arrival at the earth's surface; what should be the behavior of $t''(\theta)$ at the minimum distance? The question bears on the expected amplitude of the given wave at this minimum distance. Considering the path in the core leads to the same singularity as in the simple plane case just described. Dix [69] has shown that the second derivative should vanish except for certain highly singular velocity distributions.

Some large scale diffraction phenomena occur. The core of the earth, which has a relatively sharp boundary at a depth of 2900 kilometers, does not transmit transverse waves; and at the boundary the velocity of longitudinal waves drops from about 13 km/sec outside to about 8 km/sec inside; this is the only noteworthy exception to the rule of increasing velocity with depth. There results a shadow zone for both types of waves, within which disturbances diffracted around the core are found on the seismograms of sensitive instruments recording great earthquakes. Further, there is an angle of minimum deviation for the principal longitudinal waves refracted through the core; this angle determines the outer limit of the shadow zone, which accordingly exhibits the phenomenon known in optics as a caustic, with a focussing effect producing large amplitudes in a small range of θ. Waves apparently continuous with the others are observed irregularly within the outer limit. For years these were attributed to diffraction associated with the caustic—more precisely, with its continuation as a caustic surface within the earth; but no exact theory was attempted until quite recently [70]. Meanwhile, it has appeared that the observations are better explained by the existence of an "inner" core, the boundary of which need not be very sharp [71–73].

Many generalizations have been aimed at extending the applicability of the theory. Thus, we have handled "optical" seismology assuming a spherical earth. The effects of ellipticity are definitely observable; indeed, it has been suggested [74, p. 231] that seismological data would suffice for a rough determination of the oblateness of the terrestrial spheroid. A good beginning has been made in these problems [75–79], which fall into three evident stages: a figure-of-the-earth problem to indicate the probable variation in ellipticity with depth for the homogeneous shells constituting the earth; a solution for the propagation of elastic waves through such a body; and a detailed examination of this solution for new phenomena of reflection and refraction not possible with spherical symmetry.

Love [17] and later writers have investigated the effect of gravity on the propagation of seismic waves; this proves to be entirely negligible.

A difficult series of papers by Uller [80] offers a generalization of the wave concept as such, with seismological applications. Further critical discussion is needed. Summaries have been given by Macelwane [3, pp. 110–114) and Gutenberg [1, vol. 4, pp. 136–146].

Some extension of theory is required to account for the frequent observation that the prevailing periods of all types of seismic waves increase with distance. This cannot wholly be due to the normal spreading of an impulsive disturbance with filtering out of the shorter periods. For surface waves, where very long paths are observable, often up to several complete circuits of the earth, such an explanation can be ruled out. Surface waves show dispersion, the theory showing that velocity should vary with period; but this cannot account for the increase with distance of the period of a clearly identifiable wave group. For body waves the corresponding evidence is less conclusive, but convincing to anyone familiar with the data. The required effect will be given by some type of viscosity, or internal friction. Sezawa [81–83] has set up a theory which partially covers the case; the problem has been treated by Jeffreys [84, 85; 2, pp. 265–266], and by Gutenberg and Schlechtweg [86].

Confusion, especially when there is insistence on carrying on the discussion in apparently non-mathematical terms, arises from the use of such terms as "viscosity" and "strength" in different senses. Definition of viscosity for solids is difficult, depending on the form of the hypothetically modified equations of elasticity. The property most closely corresponding to the well known and well defined viscosity of fluids is commonly termed internal friction; the entirely different quantity which measures the slow yielding of solids under long-continued pressure is more usually termed viscosity. For discussion see Gutenberg [4, chap. XV].

The classical theory of elasticity requires the components of stress to be linear functions of those of strain. This first-order approximation is being extended by second-order theories. To this end Murnaghan's theory of finite strain [87] may be applied, assuming "constants" to be modified in a medium under strain, the case of most importance being that of hydrostatic pressure. On this basis Birch [89] has found close agreement between observed and calculated seismic wave velocities in the earth, and has shown [88] that initial hydrostatic pressure does not affect the laws of elastic wave propagation. Biot [90] has developed a general second-order theory, and has recently [91] ap-

plied it to the general problem of propagation of elastic waves when there is initial stress; hydrostatic pressure produces no effect, but other types of initial stress may lead, among other effects, to a coupling between longitudinal and transverse waves. The non-effectiveness of hydrostatic pressure restricts the applicability of the old results of Rudzki [92], who assumed the earth to be aeolotropic, with different wave velocities horizontally and vertically. He arrived at the same coupling of longitudinal and transverse waves; but his theory gave no ready means of estimating the magnitude of the supposed seismological effect.

Mathematical seismology is much in need of synthesis. Techniques to solve particular problems have been imported from other applications or from pure mathematics, without much examination of their relation to each other and to the general fields in which they originate. Hence the implications of such special results as the cubic equation characterizing the Rayleigh wave, or the Abelian integral equation in the velocity problem, are often imperfectly understood by those who are constantly working with them. Not rarely this leads to difficulties when the problems are to be generalized. Much spade work remains in exploration of the various subjects suggested by the natural articulation of the data; but the ultimate outcome should be a comprehensive treatment akin to the great treatises on optics and acoustics. To repeat—the contents of those treatises have only a limited application in seismology, not merely because of the complication in combining the mathematical characteristics of optics and acoustics, but largely because the principal observed data are not frequencies and intensities, but actual times of propagation of seismic waves.

It would be ungracious to close without reference to the interesting address by Professor Cairns [6], who has since been a welcome and stimulating visitor at the Pasadena laboratory.

In preparing for publication, much use has been made of the valuable summary by Blake [7]; his discussion and references cover a number of points not included here. The paper by Byerly [8] contains further supplementary material.

The following bibliography is selected for general usefulness or for mathematical interest. Many important papers, particularly those reporting observations, are omitted. Less than justice is done to the contributions by Jeffreys; more will be found in the Geophysical Supplement to the Monthly Notices of the Royal Astronomical Society. (Each volume of the Supplement covers a number of years; dates given are those of publication.) Further papers by Sezawa or by Sezawa and Kanai, as well as by other Japanese authors, will be

found in the Bulletin of the Earthquake Research Institute of Tokyo Imperial University. Japanese papers often are of mathematical interest only, by reason either of initial assumptions too remote from physical fact, or of failure to carry the calculations far enough for comparison with observations. All those cited here are in English. Throughout the bibliography, the language of the title as given is that of the original paper.

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BALCH GRADUATE SCHOOL OF THE GEOLOGICAL SCIENCES
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SOME SIMPLE DIFFERENTIAL DIFFERENCE EQUATIONS AND THE RELATED FUNCTIONS

H. BATEMAN

1. **Introduction.** The work of Brook Taylor, Jean Bernoulli and Daniel Bernoulli led to the formulation of differential difference equations which are all included in the equation

$$(1.1)\qquad Md^2u_n/dt^2 + 2Kdu_n/dt + Su_n = (na + a + b)(u_{n+1} - u_n) - (na + c)(u_n - u_{n-1})$$

in which M, K, S, a, b, c are constants. This equation may be treated in at least six different ways, which may be described briefly as follows:

(1) The method of simple solutions in which the aim is to express the desired solution as a sum of simple solutions of type

$$(1.2)\qquad u_n(t) = U_n(p) \exp(ipt),$$

in which p is a constant which may assume a certain set of values. This method was used with great success by the writers just named and was much improved by L. Euler, J. le Rond d'Alembert, J. L. Lagrange and J. Fourier. The method has been greatly developed during the last one hundred fifty years. Some idea of this development may be derived from the excellent report of H. Burkhardt[1] on expansions in series of oscillating functions.

(2) The method of generating functions in which a differential equation is formed for the generating function

$$G(z, t) = \sum z^n u_n(t).$$

This method may, perhaps, be associated with the names of Lagrange and Laplace as these writers developed a theory of generating functions. The important developments for the differential difference equation came quite late and began, perhaps, with the work of Koppe[2] on the function which I shall call the *influence function* for

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[1] H. Burkhardt, *Entwicklungen nach oscillirenden Funktionen und Integration der Differentialgleichungen der mathematischen Physik*, Jber. Deutschen Math. Verein. vol. 10 (1908) pp. 1–804.

[2] M. Koppe, *Die Ausbreitung einer Erschütterung an der Wellenmaschine durch einen neuen Grenzfall der Besselschen Functionen*, Program Andreas-Real-gymnasium, Berlin, No. 96 (1901) 28 pp. See also T. H. Havelock, *On the Instantaneous propagation of disturbance in a dispersive medium*, Philosophical Magazine (6) vol. 19 (1910) pp. 160–168; E. Schrödinger, *Dynamik elastischer gekoppelte Punktsysteme*, Annalen

the equation

$$d^2u_n/dt^2 = u_{n+1} + u_{n-1} - 2u_n. \tag{1.3}$$

(3) The method of definite integrals in which the aim is to express $u_n(t)$ in the form of a definite integral such as

$$u_n(t) = \int z^n f(z, t)dz.$$

(4) The operational method which depends upon the formation of a difference equation satisfied by the integral

$$U_n(x) = \int_0^\infty e^{-xt}u_n(t)dt.$$

Integrals of this type were used by L. Euler[3] and P. S. Laplace[4] to solve differential equations and difference equations by means of definite integrals. The operational method is really the inverse of the transformation of Euler and Laplace since the equation satisfied by $u_n(t)$ is known, while in the method of transformation the equation satisfied by $u_n(t)$ is derived from an equation satisfied by $U_n(x)$. A form of the operational method was used by Poisson[5] in the solution of two differential difference equations occurring in the theory of the conduction of heat in a bar.

In the case of equation (1.1) the equation for $U_n(x)$ is

$$\begin{aligned}(na + a + b)(U_{n+1} - U_n) - (na + c)(U_n - U_{n-1}) \\ + Mu_n'(0) + (2K - Mx)u_n(0) = (Mx^2 - 2Kx + S)U_n.\end{aligned} \tag{1.4}$$

This is a difference equation. It is homogeneous for $n \neq m$ if the initial conditions are of type

$$u_n(0) = 0, \quad n \neq m, \qquad u_m(0) = 1, \qquad u_n'(0) = 0.$$

For $n = m$ the equation is not homogeneous. The solution of the set

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der Physik (4) vol. 44 (1914) pp. 916–934; F. Pollaczek, *Über die Fortpflanzung mechanischer Vorgänge in einen linearer Gitter*, Annalen der Physik (5) vol. 2 (1929) pp. 991–1011.

[3] L. Euler, *De constructione aequationum*, Akademiia Nauk, Novi Commentarii vol. 9 (1737) 1744 pp. 85–97; *Integralrechnung* vol. 2 section II, chaps. 10, 11.

[4] P. S. Laplace, *Mémoire sur les approximations des formules qui sont fonctions de très grands nombres*, Mémoires de l'Académie Royale des Sciences de Paris (1782; 1785); *Oeuvres*, vol. 10.

[5] S. D. Poisson, *Mémoire sur la distribution de la chaleur dans les corps solides*, J. École Polytech. vol. 12 (1823) pp. 1–144.

of equations may be made unique by adding suitable end conditions. In some cases the condition that $U_n(x)$ should be finite for $n = \pm\infty$ is appropriate. Thus for the equation corresponding to (1.3) the *influence function* is

$$U_{m,n}(x) = (1 + x^2)^{-1/2}[(1 + x^2)^{1/2} - x]^{2n-2m}, \qquad n > m.$$

For $n < m$ the index is $2m - 2n$. This function $U_{m,n}$ corresponds to Koppe's influence function

$$u_{m,n}(t) = J_{2(n-m)}(2t),$$

and it seems desirable to use the same name in both cases. Recurrence relations which are all of the same type except for one or two values of n are of frequent occurrence in the theory of structures. A case which bears some resemblance to the present one occurs in the work of E. H. Bateman[6] on the stability of tall building frames. The framework considered by this writer is described as a two-column bent of N stories rigidly connected together, the beams being connected to the columns by joints of variable rigidity. This framework is acted on by a horizontal load at any panel point. The cases of vertical loading on the beams is also considered, the two cases of horizontal and vertical loading being discussed separately.

An influence function for equation (1.1) may be found by the method of the generating function. When the differential difference equation is of the first order the requirement that $u_n(0) = 0$, $n = m$, $u_m(0) = 1$, implies that $G(z, 0) = z^m$ and so the arbitrary factor in the solution of the differential equation for G is determined. The influence function for some simple cases is given in the following table:

1. $du_n/dt = u_{n+1} - u_n$, $t^{m-n}e^{-t}/\Gamma(m-n+1)$,
2. $du_n/dt = u_{n-1} - u_n$, $t^{n-m}e^{-t}/\Gamma(n-m+1)$,
3. $2du_n/dt = u_{n+1} - u_{n-1}$, $J_{m-n}(t)$,
4. $2du_n/dt = u_{n+1} + u_{n-1} - 2u_n$, $e^{-t}I_{m-n}(t)$,
5. $2du_n/dt = u_n - 2u_{n+1} + u_{n+2}$, $e^{t/2}(it^{1/2})^{m-n}H_{m-n}(it^{1/2})/\Gamma(m-n+1)$,

where in the last case

$$\exp(zt - t^2/2) = \sum_{n=0}^{\infty} t^n H_n(z)/n!.$$

$J_n(z)$ denotes the Bessel function of order n, $I_n(x)$ is the Bessel func-

[6] E. H. Bateman, *The stability of tall building frames*, Institute of Civil Engineers, London, Selected Paper No. 167, 1934, 49 pp.

tion of imaginary argument multiplied by the proper factor, $H_n(z)$ is the polynomial of Hermite and $\Gamma(z)$ is the Gamma function.

When only the second derivative appears on the left-hand side of the equation the requirement chosen for the definition of the influence function is $u_n'(0)=0$, $u_n(0)=0$, $n\neq m$, $u_m(0)=1$, and two additional cases of interest are:

6. $d^2u_n/dt^2=u_{n+1}-u_n$, $(t/2)^{m-n+1/2}J_{m-n-1/2}(t)/\Gamma(m-n+1)$,

7. $d^2u_n/dt^2=u_{n-1}-u_n$, $(t/2)^{n-m+1/2}J_{n-m-1/2}(t)/\Gamma(n-m+1)$.

(5) The method of power series. In the case of equation 2, for instance, there is a solution of type

$$u_n(t) = f(n) + (t-t_0)[f(n-1)-f(n)] + \cdots$$
$$= \sum_{m=0}^{\infty} (t-t_0)^m \Delta^m f(n)/m!, \qquad \Delta f(n) = f(n-1)-f(n),$$

as noted by G. Doetsch.[7] The function $f(n)$ is given by the equation $f(n)=u_n(t_0)$, and $\Delta^m f(n)=u_n^{(m)}(t_0)$. Doetsch remarks that when $u_n(t_0)=\psi_0(n,\ t)=e^{-t_0}t_0^n/n!$ we have $u_n^{(m)}(t_0)=(d^m/dt_0^m)\psi_0(n,\ t_0)=\psi_m(n,\ t_0)=p_m(n,\ t_0)\psi_0(n,\ t_0)$, where $p_m(x,\ t)$ is Charlier's polynomial. In this case $u_n(t)=\psi_0(n,\ t)$, and $u_n^{(m)}(t)=\psi_m(n,\ t)$ is also a solution.

(6) The method of successive approximations. In this method the equation is divided into two parts. One of these is multiplied by a parameter k and a solution is sought which can be expressed as a power series in k. The coefficients are determined by a succession of equations which in many cases can be solved very easily. The parameter k is then set equal to 1 after all the operations. In the case of equation 1, for instance, we use the form $u_{n+1}=k(1+d/dt)u_n$ and find that the power series for $u_n(t)$ reduces to a single term $u_n(t)=k^n(1+d/dt)^n f(t)$, where $f(t)$ is an arbitrary function which can be differentiated an unlimited number of times. In the case of equation 6 there is a corresponding solution $u_n(t)=(1+D^2)^n f(t)$ where $D=d/dt$. By choosing $f(t)=t^{-1}\cos t$ we obtain the solution $u_n(t)=-(1/2)\Gamma(n+1)\Gamma(1/2)(t/2)^{-n-1/2}Y_{n+1/2}(t)$, and by choosing $f(t)=t^{-1}\sin t$ we get the solution $u_n(t)=(1/2)\Gamma(n+1)\Gamma(1/2)(t/2)^{-n-1/2}J_{n+1/2}(t)$, the expressions for the Bessel functions being essentially those of Hargrave and Macdonald.[8]

2. **Applications of differential difference equations.** The polynomial

[7] G. Doetsch, *Die in der Statistik seltener Ereignisse auftretenden Charlierschen Polynome und eine damit zusammenhängende Differentialgleichung*, Math. Ann. vol. 109 (1934) pp. 257–266.

[8] See E. T. Whittaker and G. N. Watson, *Modern analysis*, chap. 17, Ex. 30.

of Jacques Bernoulli $B_n(x)$ satisfies the simple equation

$$d/dx\, B_n(x) = nB_{n-1}(x) \tag{2.1}$$

which is satisfied also by all polynomials of Appell's type. When $B_n(x)/n!$ is denoted by $u_n(x)$ the equation is

$$du_n/dx = u_{n-1}. \tag{2.2}$$

This is a particular case of the equation

$$du_n/dt = k_{n-1}u_{n-1} - k_nu_n \tag{2.3}$$

which occurs in the theory of probability and in the theory of the chain processes of chemistry and radioactivity. The literature on this branch of the subject is now quite large. Equation 2 of §1 is also a special case of equation (2.3). Equation 1 may be regarded as the adjoint of equation 2 and it should be noticed that the influence function for this equation is derived from the influence function for equation 2 by interchanging m and n. This property is analogous to one which occurs in the theory of the Green's function for a linear differential equation and in the theory of the influence functions of the theory of structures.

In mechanics and in the theory of elasticity the differential difference equations of chief importance are those in which central differences occur. In his interesting book *Weather prediction by numerical process*, L. F. Richardson emphasizes the advantage of using central differences when difference equations are to be used to furnish approximate solutions of differential equations occurring in the solution of mechanical problems. He was led to this conclusion by some preliminary work on the elastic problem of the masonry dam. An account of this work is given in his paper *The approximate arithmetical solution by finite differences of physical problems*, Transactions of the Royal Society of London vol. 210A (1910) p. 307.

Following Richardson's lead an attempt will be made here to use the method of divided differences in the solution of various elastic problems. For brevity differential difference equations may be called $D\Delta$-equations. This notation, however, may be thought to denote simply the equations of mixed differences studied by Poisson, Biot and others. A simple case of such an equation

$$u'(x) = au(x - \theta) - bu(x)$$

is included in an equation which has become important in economics and has been discussed in several of the recent volumes of the publication Econometrica.

3. DΔ-equations in the theory of the compound pendulum. In the case of a pendulum consisting of a light string with equal masses equally spaced and a large mass at the end, the equation of small vibrations is of type

$$m\ddot{x}_n + Sx = (M + nm)g(x_{n+1} - x_n) - (M + nm - m)g(x_n - x_{n-1}). \tag{3.1}$$

For greater generality a spring has been added so that each mass when displaced is acted upon by a restoring force proportional to the displacement x_n.

In the case of a periodic displacement $x_n(t) = X_n \cos(pt)$ the difference equation satisfied by X_n is of type

$$(n + s + 1)X_{n+1} + (n + s)X_{n-1} = (2n + 1 + r - kp^2)X, \tag{3.2}$$

where r and k are positive constants.

The equation of motion of the mass M is simply

$$M\ddot{x}_0 + Sx_0 = Mg(x_1 - x_0)$$

and so if $x_0 = X_0 \cos(pt)$ the equation connecting X_1 and X_0 is

$$(s + 1)X_1 = (1 + r - s - kp^2)X_0.$$

If $kp^2 = z$ the quantity X_n is a polynomial of degree n in z.

In the more general case in which the spring attached to the mass M is of different strength S', the equation connecting X_1 and X_0 is of type

$$(s + 1)X_1 = (q - kp^2)X_0, \tag{3.3}$$

where q is a constant and X_n is again a polynomial of degree n in z.

When the string is regarded as of infinite length, the solution of the initial value problem may be solved by finding a weight function $w(z)$ such that

$$\int_0^\infty w(z)X_n(z)X_{n'}(z)dz = \begin{cases} 0, & n' \neq n, \\ 1, & n' = n, \end{cases} \tag{3.4}$$

for then if $x_n(0) = 0$, $n \neq n'$, $x_{n'}(0) = 1$, $x_n'(0) = 0$, we have the formula for the influence function[9]

$$x_{n',n}(t) = \int_0^\infty w(z)X_n(z)X_{n'}(z)\cos(pt)dz = x_{n,n'}(t). \tag{3.5}$$

[9] The function $x_{n',n}(t)$ has also been called a *selective function*, see H. Bateman, *Selective functions and operations*, Amer. Math. Monthly vol. 41 (1934) pp. 556–562.

The function $w(z)$ often may be found by solving the problem of the moments. Since X_0 is a constant which may be taken to be unity the first equation is

$$\int_0^\infty w(z)dz = 1.$$

The recurrence relation may be used to find $X_1, X_2, \cdots$, successively, and as these are polynomials in z the orthogonal relation, with $n'=0$, will give the values of the integral

$$\int_0^\infty z^n w(z)dz, \qquad n = 1, 2, 3, \cdots.$$

In most cases of physical interest the function $w(z)$ is positive for values of z exceeding a certain number a which is either zero or a positive quantity. In the simple case of a pendulum without springs and without the mass M at the end, the function $w(z)$ is $\exp(-z)$ and $X_n=L_n(z)$ where $L_n(z)$ is the polynomial of Daniel Bernoulli,[10] J. L. Lagrange and E. Laguerre. When the pendulum is suspended from a point at which a mass would ordinarily be concentrated the number p is given by an equation of type $L_N(z)=0$, and the initial value problem may be solved by a method indicated by J. L. Lagrange[11] and Bottema[12] in which the first essential step is the derivation of an orthogonal relation of type

$$\sum_{n=0}^{N} L_n(z)L_n(z') = 0, \qquad z \neq z', \tag{3.6}$$

where z and z' are different roots of the equation $L_N(z)=0$. The displacement $u_n(t)$ is then expressed in the form

$$u_n(t) = \sum_{m=1}^{N} [c_m L_n(z_m) \cos(p_m t) + b_m L_n(z_m) \sin(p_m t)], \tag{3.7}$$

the coefficients b_m and c_m being derived by means of the orthogonal relation. The initial value problem may be solved also by means of the method used for the infinite string if the differential equation is

[10] D. Bernoulli, Akademiia Nauk, Novi Commentarii vol. 6 (1734) p. 108, vol. 7 (1935) p. 162.

[11] J. L. Lagrange, *Méchanique analitique*, Part 2, Paris, 1788; *Oeuvres*, vol. 11, Paris, 1888, section 6. See also *Miscellanea Taurinensis*, vol. 3 (1762–1765), *Oeuvres*, vol. 1, p. 534. Lord Rayleigh, *Theory of sound*, vol. 1 (1877) p. 129.

[12] O. Bottema, *Die Schwingungen eines zusammengesetzten Pendels*, Jber. Deutschen Math. Verein. vol. 42 (1933) pp. 42–60.

used to continue the values of the initial displacements and velocities to the fictitious masses beyond the point of suspension of the string. This method is generally laborious and may not give simple expressions for the displacement and velocity of the fictitious particles far away from the point of suspension.

It should be remarked that when the solution for initial velocities without initial displacement has been found, the corresponding solution for initial displacements without initial velocities may be derived by differentiation with respect to t in the manner described by the rule of Stokes for the initial value problem for a differential equation of the second order with respect to the time. d'Alembert's solution of the equation $d^2u/dt^2=d^2u/dx^2$, $u=2^{-1}f(x+t)+2^{-1}f(x-t)+(1/2)\int_{x-t}^{x+t}g(z)dz$, furnishes a good example for the explanation of the rule. The integral represents the solution of the initial value problem $u=0$, $du/dt=g(x)$ and when it is differentiated with respect to t it gives the function $2^{-1}g(x+t)+2^{-1}g(x-t)$, which is the solution of the initial value problem $u=g(x)$, $du/dt=0$. Replacing g by f in this expression and adding to the integral we have the solution of the general initial value problem $u=f$, $du/dt=g$.

4. **Derivation of asymptotic forms.** The operational method is often useful for the derivation of an asymptotic form of the influence function by means of a method developed by A. Haar,[13] G. Doetsch[14] and others. This method is the analogue for integrals of the method of Darboux for finding an estimate for large values of n of the coefficient of z^n in the expansion of a function in a power series with a finite radius of convergence. The method of Darboux is based on a study of the behavior of the function around the singularities on the circle of convergence. In the case of an integral of Laplace's type a straight line parallel to the imaginary axis is the boundary of the region of convergence and it is necessary to consider the form of the function represented at the singularities which lie on this line. For example, in the case of the integral

$$(4.1)\qquad \int_0^\infty e^{-xt}J_n(t)dt = (1+x^2)^{-1/2}[(1+x^2)^{1/2}-x]^n,\quad R(x)\geqq 0,$$

there are singularities at $x=\pm i$ on the line $R(x)=0$ and it is readily found that if $s=2n+1$

[13] A. Haar, *Über asymptotische Entwicklungen von Funktionen*, Math. Ann. vol. 96 (1926) pp. 69–107.

[14] G. Doetsch, *Ein allgemeines Prinzip der asymptotischen Entwicklung*, J. Reine Angew. Math. vol. 167 (1932) pp. 274–293.

$$J_n(t) \sim (2\pi t)^{-1/2}(e^{(it-in\pi/4)} + e^{(in\pi/4-it)})$$
$$= (2/\pi t)^{1/2} \cos [t - (n + 1/2)\pi/2].$$

Similarly the relation

$$\int_0^\infty e^{-xt}dt \int_0^t J_n(z)dz = x^{-1}(1 + x^2)^{-1/2}[(1 + x^2)^{1/2} - x]^n, \quad R(x) \geqq 0,$$

gives the estimate

$$\int_0^t J_n(z)dz \sim 1 + (2\pi t)^{-1/2} \sin [t - (n + 1/2)\pi/2].$$

These estimates are useful in the discussion of the solution of equation (1.3) which is given by the series

$$(4.2) \quad u_n(t) = \sum_{m=-\infty}^{\infty} J_{2(n-m)}(2t)u_m(0) + \sum_{m=-\infty}^{\infty} u'_m(0) \int_0^t J_{2(n-m)}(2s)ds.$$

When t is large each term in the second series gives a finite residual disturbance. These residual disturbances would be eliminated eventually in the case of a finite string by reflections at the ends. The existence of a residual displacement is worth noting, however, as in this respect equation (1.3) differs from its generalization to 2 or more variables of type n.

When only one particle has an initial displacement or velocity it is necessary, as Koppe points out, to find an estimate of $J_{2n}(2t)$ and its integral for values of $2n$ and $2t$, which are of about the same order of magnitude. Some results of this nature were already known but Koppe indicated the advantage of making a substitution of type $x-n=v(n/6)^{1/3}$ when discussing the form of $J_n(x)$ for large values of n and x.

The investigations relating to the asymptotic form of $J_n(x)$ began when Lagrange published his solution of Kepler's problem in the form of a sine series with coefficients of the form $J_n(ne)$ and analogous series were obtained by Bessel, Poisson, Hansen and others. Asymptotic forms of the coefficients were needed to decide the question of convergence and notable work was done by Laplace, Poisson, Carlini, Jacobi, Cauchy and Hansen. A second period of investigation began when Debye, Watson and others used the method of steepest descent to estimate the values of definite integrals representing Bessel functions, and methods based on the transformation of differential equations and integrals were employed by Nicholson, Langer, Van Veen, Fock and other writers. Estimates of the integral of the function

$J_n(x)$ have been given by Pollaczek. Some of the results of these investigations are summarized below, the chief terms of the estimates are the only ones given.

$$x = n \operatorname{sech} a, \qquad a > 0,$$

$$J_n(x) \sim (2\pi n \tanh a)^{-1/2} e^{-n(a-\tanh a)},$$

$$\int_0^x J_{n-1}(w)dw \sim (2\pi n \tanh a)^{-1/2}(\coth a + 1)e^{-n(a-\tanh a)},$$

$$x = n \sec b, \qquad 0 \leqq b \leqq \pi/4,$$

$$J_n(x) \sim (1/2)\pi n \tan b)^{-1/2} \cos [n(\tan b - b) - \pi/4],$$

$$\int_0^x J_{n-1}(w)dw \sim 1 + ((1/2)\pi n \tan b)^{-1/2} \operatorname{cosec} b \cdot \cos [n(\tan b - b) + b - 3\pi/4],$$

$$J_n(x) \sim (1 - b \cot b)[\cos (\pi/6)J_{1/3}(n \tan b - nb) - \sin (\pi/6)Y_{1/3}(n \tan b - nb)].$$

If $|x-n| \ll n^{2/5}$ Nicholson's estimate is

$$J_n(x) \sim (6/x)^{1/3} Ai[(6/x)^{1/3}(x - n)], \qquad Ai(w) = \int_0^\infty \cos (t^3 - wt)dt,$$

and when $x-n=v(n/6)^{1/3}$ Pollaczek's estimate is

$$\int_0^x J_{n-1}(w)dw \sim (1/2\pi i) \int_C e^{-t^3+vte^{i\pi/3}}dt/t.$$

For these results reference may be made to Burkhardt's report,[1] Watson's book,[15] Pollaczek's paper[2] and the papers of Langer[16] and Fock.[17]

The following table shows that the maximum of $J_{2n}(2t)$ occurs at a time slightly greater than $t=n$.

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[15] G. N. Watson, *Bessel functions*, Cambridge University Press, 1922.

[16] R. E. Langer, *On the asymptotic solutions of ordinary differential equations with an application to the Bessel functions of large order*, Trans. Amer. Math Soc. vol. 33 (1931) pp. 23–64; *On the asymptotic solutions of ordinary differential equations, with an application to the Bessel functions of large complex order*, Trans. Amer. Math. Soc. vol. 34 (1932) pp. 447–480.

[17] V. Fock, *Neuer asymptotischer Ausdruck für Besselsche Funktionen*, C. R. (Doklady) Acad. Sci. U.R.S.S. (2) vol. 1 (1934) pp. 97–102; see also A. Svetlov, *Ueber die asymptotischen Ausdrücke für Besselsche Funktionen bei grossen Indexen*, ibid. (2) vol. 2 (1934) p. 448.

t	0	1	2	3	4	5	6
n							
0	1	.2239	−.3971				
1	0	.3528	.3641	−.2429	−.1130		
2		.0340	.2811	.3576	−.1054		
3			.0491	.2458	.3376	−.0145	
4				.0565	.2235	.3179	
5					.0608	.2075	.3005

5. **Surges in springs and connected systems of springs.** The use of concentrated masses has been found advantageous in the study of surges in springs such as those of spiral or helical type. Reference may be made to the bibliography in an admirable paper on the subject by K. J. De Juhasz.[18]

An important equation which is a simple generalization of (1.3) is obtained when the concentrated masses on a light string are mounted on springs arranged either along a straight line or on the circumference of a circle or helix. The simplest form of this equation is

$$d^2u_n/dt^2 = u_{n+1} + u_{n-1} - 2(1+k)u_n, \tag{5.1}$$

where k is a positive constant. If $G(z,t) = \sum_{n=-\infty}^{\infty} z^n u_n(t)$ is the generating function $d^2G/dt^2 = (z^{-1}+z-2k-2)G$ and if $u_n(0)=0$, $n\neq 0$, $u_0(0)=1$ $u_n'(0)=0$, an appropriate solution is

$$G(z,t) = ch[t(z^{-1}+z-2k-2)^{1/2}]. \tag{5.2}$$

It is readily seen that $\int_0^\infty \exp(-xt)G(z,t)dt = x(x^2+2k+2-z-z^{-1})^{-1}$. For sufficiently large values of $R(x)$

$$\int_0^\infty e^{-xt}u_n(t)dt = x[(x^2+2k+2)^2-4]^{-1/2}[k+1+x^2/2-y]^n, \tag{5.3}$$

where $y^2=(k+1+x^2/2)^2-1$.

An application of Haar's method indicates that for large positive values of t

$$u_n(t) \sim (2\pi t)^{-1/2}[(2k)^{1/4}\cos\{t(2k)^{1/2}+\pi/4\} + (2k+4)^{1/4}\cos\{t(2k+4)^{1/2}-\pi/4-n\pi\}]. \tag{5.4}$$

It should be noticed that we also have the equation

[18] K. J. De Juhasz, *Graphical analysis of surges in mechanical springs*, Journal of the Franklin Institute vol. 226 (1938) pp. 505–526, 631–644. The free vibrations of helical springs are treated by the same author in vol. 227 (1939) pp. 647–672.

$$(5.5)\qquad \int_0^\infty e^{-ut^2}u_n(t)dt = (1/2)(\pi/u)\exp(-(k+1)/2u)I_n(1/2u),$$

but this expression for the Laplace integral has an essential singularity at the point $u=0$ and is not so convenient for the deduction of the asymptotic form of $u_n(t)$. An extension of Haar's method is needed which will make the deduction easy in such a case. In applying Haar's method to equation (5.3) it is advisable to approach each of the four singularities, $x=i(2k+4)^{1/2}$, $x=i(2k)^{1/2}$, $x=-i(2k)^{1/2}$, $x=-i\cdot(2k+4)^{1/2}$, from the right along a line parallel to the real axis. The limiting forms of the arguments of the factors of $[(x^2+2k+2)^2-4]^{-1/2}$ may then be written down for each line of approach without any error.

It is important to notice that in the present case we find by putting $x=0$ in equation (5.3) that

$$(5.6)\qquad \int_0^\infty u_n(t)dt = 0.$$

Hence there is no residual displacement when all the masses are originally in the equilibrium position and the mass numbered 0 is given an initial velocity. The displacement $v_n(t)$ at time t is in fact $v_n(t)=\int_0^t u_n(s)ds$ and this tends to zero as $t\to\infty$. This result is quite different from that in Koppe's case when $\int_0^\infty J_{2n}(2s)ds=1/2$.

6. **The equations of Born and Kármán.** In their work on crystal lattices in vibration Born and Kármán[19] considered equation (1.3) and also a pair of equations which may be written in the form

$$(6.1)\qquad \begin{aligned} d^2x_{2n}/dt^2 &= a^2(x_{2n+1}+x_{2n-1}-2x_{2n}),\\ d^2x_{2n+1}/dt^2 &= b^2(x_{2n+2}+x_{2n}-2x_{2n+1}).\end{aligned}$$

If the initial conditions are $x_{2n}(0)=0$, $n\neq m$, $x_{2n+1}(0)=1$, $x_{2n+1}(0)=0$, $x_n'(0)=0$, there are two generating functions,

$$(6.2)\qquad E(z,t)=\sum_{n=-\infty}^{\infty}x_{2n}(t)z^{2n},\qquad O(z,t)=\sum_{n=-\infty}^{\infty}x_{2n+1}(t)z^{2n+1},$$

which satisfy the equations

$$(6.3)\qquad \begin{aligned} d^2E/dt^2 &= a^2(z+z^{-1})O-2a^2E,\\ d^2O/dt^2 &= b^2(z+z^{-1})E-2b^2O.\end{aligned}$$

[19] M. Born and Th. v. Kármán, *Ueber Schwingungen in Raumgittern*, Physikalische Zeitschrift vol. 13 (1912) pp. 297–309.

If $m=0$,

$$d^2 = a^2b^2(z + z^{-1})^2 + (a^2 - b^2)^2,$$

$$d - (a^2 + b^2) = c^2, \qquad d + a^2 + b^2 = k^2,$$

$$E(z, t) = [(d + b^2 - a^2)/2d]ch(ct) + [(d + a^2 - b^2)/2d]\cos(kt),$$

$$O(z, t) = [b^2(z + z^{-1})/ad][ch(ct) - \cos(kt)].$$

It is advantageous to write

$$x_{2n}(t) = u_{2n}(t, a, b) + (b^2 - a^2)v_{2n}(t, a, b), \tag{6.4}$$

$$x_{2n+1}(t) = b^2v_{2n}(t, a, b) + b^2v_{2n+2}(t, a, b),$$

where u_{2n} and v_{2n} are symmetric functions of a and b. The generating functions of $u_{2n}(t)$ and $v_{2n}(t)$ are respectively

$$U(z, t) = \sum_{n=-\infty}^{\infty} z^{2n}u_{2n}(t, a, b) = (1/2)ch(ct) + (1/2)\cos(kt), \tag{6.5}$$

$$V(z, t) = \sum z^{2n}v_{2n}(t, a, b) = (1/2d)[ch(ct) - \cos(kt)],$$

and if $U_n(u, a, b) = \int_0^\infty e^{-ut}u_n(t, a, b)dt$, $V_n(u, a, b) = \int_0^\infty e^{-ut}v_n(t, a, b)dt$,

$$U_n(u, a, b) = (u^2 + a^2 + b^2)D^nE, \qquad V_n(u, a, b) = D^nE, \tag{6.6}$$

where $2abD = (u^2 + 2a^2)^{1/2}(u^2 + 2b^2)^{1/2} - (u^4 + 2a^2u^2 + 2b^2u^2)^{1/2}$, $E = [(u^2+2a^2)(u^2+2b^2)(u^2+2a^2+2b^2)]^{-1/2}$. There is also a relation

$$\int_0^\infty e^{-ut}u_n(t, a, b)dt/t = D^n/2n. \tag{6.7}$$

Use of the multiplication theorem for integrals of Laplace's type gives the addition formula

$$\int_0^t v_m(t - s, a, b)u_n(s, a, b)ds/s = v_{m+n}(t, a, b)/2n.$$

The differential equations satisfied by the functions $u_{2n}(t, a, b)$, $v_{2n}(t, a, b)$ are

$$d^2v_{2n}/dt^2 = u_{2n} - (a^2 + b^2)v_{2n}, \tag{6.8}$$

$$d^2u_{2n}/dt^2 + (a^2 + b^2)d^2v_{2n}/dt^2 = a^2b^2(v_{2n+2} + v_{2n-2} - 2v_{2n}).$$

Elimination of u or v gives the result that u and v both satisfy the differential difference equation

$$d^4u_{2n}/dt^4 + 2(a^2 + b^2)d^2u_{2n}/dt^2 = a^2b^2(u_{2n+2} + u_{2n-2} - 2u_{2n}).$$

The behavior of u_{2n} and v_{2n} for large values of t may be found by

Haar's method. The result is that if $b>a>0$, $m=0$,

$$
\begin{aligned}
u_{2n}(a,b,t) = 2^{-3/4}(\pi t)^{-1/2}\big[(b^2-a^2)^{1/2}\big\{a^{-1/2}b^{-1} \\
\cos\left[at2^{1/2}-(2n+1/2)\pi/2\right]+a^{-1}b^{-1/2} \\
\cos\left(bt2^{1/2}-(2n-1/2)\pi/2\right)\big\}+((a^2+b^2)^{3/4}/ab) \\
\cdot\cos\left\{t(2a^2+2b^2)^{1/2}-\pi/4\right\}\big], \qquad (6.9)
\end{aligned}
$$

$$
\begin{aligned}
v_{2n}(a,b,t)\sim 2^{-3/4}(\pi t)^{-1/2}\big[(1/ab(a^2+b^2)^{1/4}) \\
\cdot\cos\left\{t(2a^2+2b^2)^{1/2}-5\pi/4\right\}+(1/ab^{1/2}(b^2-a^2)^{1/2}) \\
\cdot\cos\left\{t(2b^2)^{1/2}-3\pi/4-n\pi\right\}+(1/b(a)^{1/2}(b^2-a^2)^{1/2}) \\
\cdot\cos\left\{t(2a^2)^{1/2}-\pi/4-n\pi\right\}\big].
\end{aligned}
$$

Asymptotic forms for $x_{2n}(t)$ and $x_{2n+1}(t)$ were worked out by one of my students, C. C. Lin, for the case $b>a$, with the aid of expressions for $\int_0^\infty e^{-ut}x_{2n}(t)dt$, $\int_0^\infty e^{-ut}x_{2n+1}(t)dt$. The results agree with those derived from the foregoing formulae. The case $a>b>0$ seemed more difficult but the difficulties are avoided in the present method, as use is made of the symmetry of u_{2n} and v_{2n} in a and b. Lin's result is that if $b^2>a^2$, $c=[2(a^2+b^2)]^{1/2}$,

$$
\begin{aligned}
x_{2n}(t) &\sim (2/\pi t)^{1/2}\big[(a/b(c)^{1/2})\cos(ct-\pi/4) \\
&\quad+((b^2-a^2)^{1/2}/b(a(2)^{1/2})^{1/2})\cos(a(t)^{1/2}-\pi/4-n\pi+m\pi)\big], \\
x_{2n+1}(t) &= (2b/a)(2/\pi ct)^{1/2}\cos(ct-\pi/4).
\end{aligned}
$$

Asymptotic forms are needed also for the case in which ct and $2n$ are both large and of the same order of magnitude. Differential equations for $u_{2n}(t)$ and $v_{2n}(t)$ may be derived from those satisfied by their Laplace transforms, but the equations are of high degree and the work of deriving the asymptotic forms from the differential equations has not yet been attempted.

7. **Derivation of the solution of an initial value problem with the aid of Poisson's formula for a solution of the wave-equation.** Poisson's formula[20] for a function $w(x, y, z, t)$ which satisfies the wave equation $\partial^2w/\partial t^2=k^2(\partial^2w/\partial x^2+\partial^2w/\partial y^2+\partial^2w/\partial z^2)$ is

$$
(7.1)\quad 4\pi w(x,y,z,t) = d/dt\left[t\int_0^\pi\int_0^{2\pi} dSf(kt)\right]+t\int_0^\pi\int_0^{2\pi} dSg(kt),
$$

[20] S. D. Poisson, *Mémoire pour la propagation du mouvement dans les milieux élastiques*, Mémoires de l'Académie des sciences vol. 10 (1823).

where $f(kt)$ is an abbreviation for the result of replacing a, b, c by $kt \sin\theta\cos\phi$, $kt \sin\theta \sin\phi$, $kt\cos\theta$ respectively in the function

$$f(x+a, y+b, z+c) \tag{7.2}$$

and $g(kt)$ has a similar meaning. The symbol dS stands for the elementary solid angle $\sin\theta d\theta d\phi$.

It should be noticed that the function (7.2) is a solution of the 3 partial differential equations

$$\partial^2 f/\partial x^2 = \partial^2 f/\partial a^2, \qquad \partial^2 f/\partial y^2 = \partial^2 f/\partial b^2, \qquad \partial^2 f/\partial z^2 = \partial^2 f/\partial c^2. \tag{7.3}$$

We now add subscripts p, q, r and assume the function f to be a solution of the additional equations

$$\begin{aligned} \partial^2 f_p/\partial a^2 &= f_{p+1} + f_{p-1} - 2f_p, \\ \partial^2 f_q/\partial b^2 &= f_{q+1} + f_{q-1} - 2f_q, \\ \partial^2 f_r/\partial c^2 &= f_{r+1} + f_{r-1} - 2f_r. \end{aligned} \tag{7.4}$$

The function $w(x, y, z, t)$ then becomes $w_{p,q,r}(x, y, z, t)$ and is a solution of the differential difference equation

$$\begin{aligned} d^2 w_{p,q,r}/dt^2 = k^2(&w_{p+1,q,r} + w_{p-1,q,r} + w_{p,q+1,r} + w_{p,q-1,r} \\ &+ w_{p,q,r+1} + w_{p,q,r-1} - 6w_{p,q,r}). \end{aligned} \tag{7.5}$$

The proof is quite simple because, on account of the equations (7.4), (7.3) satisfied by the function $f_{p,q,r}(x+a, y+b, z+c)$, the right-hand side of (7.5) becomes, by virtue of (7.1), the sum of two integrals the first of which has in the integrand

$$k^2(\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2 + \partial^2 f/\partial z^2) \qquad \text{(subscripts } p, q, r \text{ omitted)},$$

while the second contains a similar expression derived from the function $g_{p,q,r}(x+a, y+b, z+c)$, which is supposed to satisfy the same two sets of equations as $f_{p,q,r}$. When, however, a, b, and c have the assigned expressions involving t, θ, and ϕ it is a simple consequence of the properties of Poisson's integral that the resulting expression is equal to $\partial^2 w_{p,q,r}/\partial t^2$.

In the particular case in which

$$\begin{aligned} &f_{p,q,r}(x+a, y+b, z+c) \\ &\quad = J_{2(p-l)}(2x+2a)J_{2(q-m)}(2y+2b)J_{2(r-n)}(2z+2c) \end{aligned}$$

and $g_{p,q,r}(x+a, y+b, z+c)$ has a similar form we obtain a solution which furnishes a solution of the initial value problem for the equation (7.5). Putting $x=0$, $y=0$, $z=0$ we write

$$4\pi I_{p,q,r;l,m,n}(t) = \frac{d}{dt}\Big[t\int_0^\pi\int_0^{2\pi} \sin\theta\, d\theta d\phi J_{2(p-l)}(2kt\sin\theta\cos\phi) \cdot J_{2(q-m)}(2kt\sin\theta\sin\phi)J_{2(r-n)}(2kt\cos\theta)\Big]. \tag{7.6}$$

In the two-dimensional case in which w is independent of r the equation for $w_{p,q}$ is

$$d^2w_{p,q}/dt^2 = k^2(w_{p+1,q} + w_{p-1,q} + w_{p,q+1} + w_{p,q-1} - 4w_{p,q}), \tag{7.7}$$

and the influence function is given by

$$4\pi I_{p,q;l,m}(t) = \frac{d}{dt}\Big[t\int_0^\pi\int_0^{2\pi} \sin\theta\, d\theta d\phi J_{2(p-l)}(2kt\sin\theta\cos\phi) \cdot J_{2(q-m)}(2kt\sin\theta\sin\phi)\Big]. \tag{7.8}$$

A change of axes indicates that there is a second formula

$$4\pi I_{p,q;l,m}(t) = d/dt\Big[t\int_0^\pi\int_0^{2\pi} \sin\theta\, d\theta d\phi J_{2p-2l}(2kt\sin\theta\cos\phi) \cdot J_{2q-2m}(2kt\cos\theta)\Big].$$

The integration with respect to ϕ can be effected and we get

$$I_{p,q;l,m}(t) = \frac{1}{2}\frac{d}{dt}\Big[t\int_0^\pi \sin\theta\, d\theta J^2_{p-l}(kt\sin\theta)J_{2q-2m}(2kt\cos\theta)\Big]. \tag{7.9}$$

This is the simplest expression that has been obtained for the function I. The method of the generating function indicates that

$$x^l y^m ch\big[kt(x + x^{-1} + y + y^{-1} - 4)^{1/2}\big] = \sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty} I_{p,q;l,m}(t)x^p y^q. \tag{7.91}$$

If we write

$$(x + x^{-1} + y + y^{-1} - 4) = (x^{1/2} - x^{-1/2})^2 + (y^{1/2} - y^{-1/2})^2,$$

a power series for I may be obtained. On the other hand, if we multiply the generating function by $\exp(-zt^2)$ and integrate with respect to t from 0 to ∞, the left-hand side of equation (7.91) becomes

$$(1/2)(\pi/z)^{1/2}\exp k^2(x + x^{-1} + y + y^{-1} - 4)/4u,$$

and so

$$(7.92)\quad 2\int_0^\infty e^{-st^2} I_{p,q;l,m}(t)\,dt = (\pi/z)^{1/2} e^{-k^2/4z} I_{p-l}(k^2/2z) I_{q-m}(k^2/2z).$$

When u is large and positive $I_n(u) \sim (2\pi u)^{-1/2} e^u$ and so when $z \to 0$ we obtain the equation

$$(7.93)\quad \int_0^\infty I_{p,q;l,m}(t)\,dt = 0.$$

Now the solution of the initial value problem is

$$(7.94)\quad w_{p,q}(t) = \sum I_{p,q;l,m}(t) w_{l,m}(0) + \sum \int_0^t ds I_{p,q;l,m}(s) ds w'_{l,m}(0),$$

consequently there is no residue (as $t \to \infty$) arising from an initial velocity imparted to one of the particles, and the associated displacement of each particle ultimately approaches zero.

It may be noted that the function $I_{p,q}(t) \equiv I_{p,q;0,0}(t)$ satisfies the partial difference equation

$$(7.95)\quad q(I_{p+1,q} - I_{p-1,q}) = p(I_{p,q+1} - I_{p,q-1}),$$

which is the analogue of the equation $y\partial F/\partial x = x\partial F/\partial y$ which expresses that F depends only on x^2+y^2.

When it is desired to extend the present analysis to the equation $\partial^2 w/\partial t^2 + \partial^4 w/\partial x^4 + 2\partial^4 w/\partial x^2 \partial y^2 + \partial^4 w/\partial y^4 = 0$, which occurs in the theory of a vibrating plate, use may be made of the approximations to partial derivatives of high order given in the paper of E. Pflanz.[21] The expression which replaces the derivatives of the fourth order then consists of 25 terms. References are given to the work of other writers on this subject.

8. **Elastic waves in a lattice.** Taking first the case of motion in two dimensions we replace the usual equations of motion by differential difference equations. Let $u_{p,q}$, $v_{p,q}$ be the components of displacement at the lattice point (p, q) and let P, Q be operators such that

$$(8.1)\quad Pu_{p,q} = u_{p+1/2,q} - u_{p-1/2,q}, \qquad Qu_{p,q} = u_{p,q+1/2} - u_{p,q-1/2};$$

then if λ, μ are the elastic constants of Lamé the components of stress may be defined to be

$$(8.2)\quad \begin{aligned} \overline{xx}_{p,q} &= 2\mu Pu_{p,q} + \lambda(Pu_{p,q} + Qv_{p,q}), \\ \overline{yy}_{p,q} &= 2\mu Qv_{p,q} + \lambda(Pu_{p,q} + Qv_{p,q}), \\ \overline{xy}_{p,q} &= \overline{yx}_{p,q} = \mu(Pv_{p,q} + Qu_{p,q}), \end{aligned}$$

[21] E. Pflanz, *Über die Annäherung linearer partieller Differentialausdrücke durch finite Ausdrücke*, Jber. Deutschen Math. Verein. vol. 48 (1938) pp. 41–48.

and the equations of motion are

$$(8.3)\qquad \begin{aligned} \rho d^2u_{p,q}/dt^2 &= \mu P^2u_{p,q} + (\lambda+\mu)(P^2u_{p,q} + PQv_{p,q}) + \mu Q^2u_{p,q},\\ \rho d^2v_{p,q}/dt^2 &= \mu P^2v_{p,q} + (\lambda+\mu)(PQu_{p,q} + Q^2v_{p,q}) + \mu Q^2v_{p,q}. \end{aligned}$$

These may be satisfied by writing

$$(8.4)\qquad \begin{aligned} u_{p,q} &= A\alpha a^p b^q e^{iwt} + B\gamma a^p c^q e^{iwt},\\ v_{p,q} &= A\beta a^p b^q e^{iwt} - B\alpha a^p c^q e^{iwt}, \end{aligned}$$

where A, B, a, b, c, w are arbitrary constants and $\alpha=a^{1/2}-a^{-1/2}$, $\beta=b^{1/2}-b^{-1/2}$, $\gamma=c^{1/2}-c^{-1/2}$. Substitution in the equations of motion indicates, however, that these will be satisfied when a, b, c, w are related to ρ, λ, μ in such a way that

$$(8.5)\qquad -\rho w^2 = (\lambda+2\mu)(\alpha^2+\beta^2), \qquad -\rho w^2 = \mu(\alpha^2+\gamma^2).$$

The boundary condition will be taken to be $(\overline{yy}_{p,q})=0$, $\overline{xy}_{p,q}=0$, for $q=0$. This gives the two conditions $A[2\mu\beta^2+\lambda(\alpha^2+\beta^2)]=2\mu B\alpha\gamma$, $2A\alpha\beta=B(\alpha^2-\gamma^2)$. Elimination of A and B gives the equation

$$[2\mu\beta^2+\lambda(\alpha^2+\beta^2)](\alpha^2-\gamma^2) = 4\mu\alpha^2\beta\gamma.$$

When the values of β and γ are inserted a relevant solution of this equation is

$$(8.6)\qquad w = -iV_R\alpha,$$

where V_R is the velocity of the Rayleigh wave. If $a=e^{is}$, $b=e^{-r}$, $c=e^{-s}$ we have

$$\beta = 2sh(r/2), \quad \gamma = 2sh(s/2), \quad \alpha = 2i\sin(z/2), \quad w = \pm 2V_R\sin(z/2).$$

Also if V_L is the velocity of propagation of longitudinal waves, V_T the velocity of propagation of transverse waves,

$$(8.7)\qquad \begin{aligned} w^2/V_L^2 &= 4\sin^2(z/2) - 4sh^2(r/2),\\ w^2/V_T^2 &= 4\sin^2(z/2) - 4sh^2(s/2). \end{aligned}$$

The expressions for the displacements now take the form

$$\beta = -2sh(r/2), \gamma = -2sh(s/2), \alpha = 2i\sin(z/2), w = \pm 2V_R\sin(z/2)$$

$$(8.8)\qquad \begin{aligned} u_{p,q} &= C(z)(2-V_R^2/V_T^2)e^{i(ps\pm 2V_Rt\sin(z/2))-qr}\\ &\quad -(1/2)C(z)(2-V_R^2/V_L^2)^2 e^{i(ps\pm 2V_Rt\sin(z/2))-qs},\\ v_{p,q} &= +iC(z)(2-V_R^2/V_T^2)(1-V_R^2/V_L^2)^{1/2} e^{i(ps\pm 2V_Rt\sin(z/2))-qr}\\ &\quad +2iC(z)(1-V_R^2/V_L^2)^{1/2} e^{i(ps\pm 2V_Rt\sin(z/2))-qs}, \end{aligned}$$

where $C(z)$ is an arbitrary function of z. Taking this to be unity and integrating with respect to z between 0 and 2π we obtain a type of influence displacement, the values of $u_{p,0}$ and $v_{p,0}$ being

$$
\begin{aligned}
u_{p,0} &= (1/2)\,(V_R^2/V_T^2)\,(2 - V_R^2/V_T^2)\,[\overline{J}_{2p}(2V_R t) + i\overline{E}_{2p}(2V_R t)], \\
v_{p,0} &= (V_R^2/V_T^2)(1 - V_R^2/V_L^2)^{1/2}[\overline{E}_{2p}(2V_R t) - i\overline{J}_{2p}(2V_R t)],
\end{aligned} \tag{8.9}
$$

where

$$
\overline{J}_{2p}(x) = \frac{1}{2\pi}\int_0^{2\pi} \cos\,(p\theta \pm x \sin\theta/2)d\theta = J_{2p}(\mp x),
$$

$$
\overline{E}_{2p}(x) = \frac{1}{2\pi}\int_0^{2\pi} \sin\,(p\theta \pm x \sin\theta/2)d\theta = E_{2p}(\mp x).
$$

$J_v(x)$ is the Bessel function of order v when v is an integer and $E_v(x)$ is Weber's function. When x is large and positive the estimates given in Watson's *Bessel functions*[15] are

$$
J_v(x) \sim (\pi x/2)^{-1/2} \cos\left(x - \frac{2v+1}{4}\,\pi\right),
$$

$$
E_v(x) \sim (\pi x/2)^{-1/2} \sin\left(x - \frac{2v+1}{4}\,\pi\right).
$$

It should be noticed that $\overline{E}_{2p}(0) = 0$, $\overline{J}_{2p}(0) = 0$, $p \neq 0$, $\overline{J}_0(0) = 1$. An initial displacement of the lattice point (0, 0) leaves no residue at any surface point as $t \to \infty$.

A method of finding an estimate of $E_v(x)$ when both x and v are large is indicated in Watson's *Bessel functions*.

CALIFORNIA INSTITUTE OF TECHNOLOGY

THE APRIL MEETING IN NEW YORK

The three hundred ninety-fifth meeting of the American Mathematical Society was held at Hunter College on Saturday, April 24, 1943. The attendance included the following one hundred forty-two members of the Society:

R. L. Anderson, T. W. Anderson, R. G. Archibald, L. A. Aroian, A. V. Baez, Y. K. Bal, J. L. Barnes, A. E. Basch, P. T. Bateman, E. G. Begle, Stefan Bergman, Felix Bernstein, Garrett Birkhoff, Gertrude Blanch, Henry Blumberg, S. G. Bourne, C. B. Boyer, A. T. Brauer, A. B. Brown, R. C. Buck, Hobart Bushey, J. H. Bushey, R. H. Cameron, B. E. Carlson, T. F. Cope, A. P. Cowgill, H. B. Curry, J. H. Curtiss, M. D. Darkow, A. S. Day, R. P. Dilworth, Jesse Douglas, R. J. Duffin, Nelson Dunford, Jacques Dutka, J. E. Eaton, M. L. Elveback, Benjamin Epstein, Paul Erdös, J. M. Feld, A. D. Fialkow, W. B. Fite, Edward Fleisher, R. M. Foster, G. A. Foyle, J. S. Frame, K. O. Friedrichs, D. L. Fuller, R. E. Fullerton, H. P. Geiringer, B. P. Gill, Leonard Gillman, Wulf Gotz, Bernard Greenspan, Lewis Greenwald, C. C. Grove, W. W. Gutzman, Jacques Hadamard, B. L. Hagen, D. W. Hall, F. C. Hall, N. A. Hall, Hans-Karl Hammer, Einar Hille, Banesh Hoffmann, T. R. Hollcroft, M. W. Hopkins, E. M. Hull, Witold Hurewicz, L. C. Hutchinson, Walter Jennings, R. A. Johnson, F. E. Johnston, A. W. Jones, F. B. Jones, H. A. Jordan, Irving Kaplansky, Edward Kasner, D. E. Kibbey, Mark Kormes, Arthur Korn, B. A. Lengyel, Marie Litzinger, L. H. Loomis, E. R. Lorch, A. N. Lowan, R. K. Luneburg, L. A. MacColl, C. C. MacDuffee, H. M. MacNeille, H. F. MacNeish, D. S. Miller, L. W. Miller, E. C. Molina, C. N. Moore, F. J. Murray, C. O. Oakley, J. C. Oxtoby, M. K. Peabody, E. L. Post, M. H. Protter, A. L. Putnam, R. G. Putnam, L. L. Rauch, Mina Rees, F. G. Reynolds, Moses Richardson, R. G. D. Richardson, C. E. Rickart, J. F. Ritt, S. L. Robinson, P. C. Rosenbloom, Arthur Sard, Max Sasuly, A. T. Schafer, R. D. Schafer, I. E. Segal, James Singer, P. A. Smith, W. J. Strange, E. G. Straus, W. C. Strodt, Otto Szász, J. D. Tamarkin, J. I. Tracey, Peter Treuenfels, H. F. Tuan, Bryant Tuckerman, J. W. Tukey, D. F. Votaw, G. L. Walker, André Weil, Alexander Weinstein, Louis Weisner, R. A. Wetzel, George Whaples, A. P. Wheeler, John Williamson, Jack Wolfe, Fumio Yagi, Oscar Zariski, Antoni Zygmund.

The meeting opened at 10 A.M. in two sections, one for papers in Analysis, Professor Anna P. Wheeler presiding, and one for papers in Algebra, Geometry and Theory of Numbers, Professor Garrett Birkhoff presiding. At the general session immediately following these sections Professor Nelson Dunford gave an address entitled *Spectral theory*. Vice President C. C. MacDuffee presided at this session. In the afternoon at 2 P.M. there was a general session at which Professor R. H. Cameron gave an address entitled *Absolutely convergent trigonometric sums*. Vice President J. D. Tamarkin presided at this session.

The hospitality of Hunter College was very much appreciated by all who attended the meeting. By special arrangement, for the convenience of guests, an excellent lunch was served in the Hunter College Cafeteria.

Titles and cross references to the abstracts of the papers read follow below. Papers whose abstract numbers are followed by the letter *t* were read by title. Papers numbered 1 to 6 were read in the section for Analysis, papers 7 to 13 in the section for Algebra, Geometry and Theory of Numbers; papers 14 to 23 were read by title. Paper 11 was read by Professor Kasner. Professor Alexandroff was introduced by Professor Solomon Lefschetz, and Dr. Kober by Professor Einar Hille.

1. L. H. Loomis: *Spherical volume as a coordinate-free basis for Lebesgue measure theory.* (Abstract 49-5-143.)

2. Otto Szász: *On some trigonometric summability methods and Gibbs phenomenon.* (Abstract 49-5-148.)

3. Alexander Weinstein: *Differential equations with general boundary conditions.* (Abstract 49-5-154.)

4. Henry Blumberg: *On arbitrary point transformations.* (Abstract 49-1-25.)

5. C. N. Moore: *On the relationship between Nörlund means of a certain type.* (Abstract 49-1-37.)

6. R. J. Duffin: *Some representations for Fourier transforms.* (Abstract 49-5-132.)

7. J. E. Eaton: *A Galois theory for differential fields.* (Abstract 49-5-127.)

8. R. P. Dilworth: *Lattices with unique complements.* (Abstract 49-5-126.)

9. A. T. Brauer: *On the non-existence of odd perfect numbers of form* $p^{\alpha}q_1^2q_2^2 \cdots q_{t-1}^2q_t^4$. (Abstract 49-5-125.)

10. Jacques Hadamard: *On fractional iteration and connected questions.* (Abstract 49-5-162.)

11. Edward Kasner and John DeCicco: *Generalized transformation theory of isothermal families.* (Abstract 49-5-164.)

12. T. R. Hollcroft: *Plane curve systems with distinct nodes and cusps and of negative virtual dimension.* (Abstract 49-5-163.)

13. Jacques Dutka: *Transversality in higher space.* (Abstract 49-5-161.)

14. Paul Alexandroff: *On homological situation properties of complexes and closed sets.* (Abstract 49-7-198-*t*.)

15. H. R. Branson: *On the difference equation of a general quantum mechanical problem.* (Abstract 49-5-155-*t*.)

16. Nelson Dunford: *Spectral theory.* I. *Convergence to projections.* (Abstract 49-7-185-*t*.)

17. Edward Kasner and John DeCicco: *The congruence of element-series associated with a polygenic function.* (Abstract 49-5-165-*t*.)

18. Herman Kober: *On the approximation to integrable functions by integral functions.* (Abstract 49-7-187-*t*.)

19. E. R. Lorch: *The theory of analytic functions in normed abelian vector rings.* (Abstract 49-7-188-*t*.)

20. A. N. Lowan and H. E. Salzer: *Coefficients in the expansion of derivatives in terms of central differences.* (Abstract 49-7-196-*t*.)

21. Robert Schatten: *On certain properties of crossnorms.* (Abstract 49-5-146-*t*.)

22. A. E. Taylor: *Banach spaces of functions representable by Cauchy's integral formula.* (Abstract 49-5-149-*t*.)

23. L. I. Wade (National Research Fellow): *Transcendence properties of the Carlitz ψ-function.* (Abstract 49-5-130-*t*.)

T. R. HOLLCROFT,
Associate Secretary

THE APRIL MEETING IN CHICAGO

The three hundred ninety-sixth meeting of the American Mathematical Society was held at the Museum of Science and Industry, Chicago, Illinois, on Friday and Saturday, April 23-24, 1943. The attendance was about one hundred fifty, including the following one hundred and nineteen members of the Society:

Leon Alaoglu, A. A. Albert, B. H. Arnold, Emil Artin, W. L. Ayres, Reinhold Baer, R. H. Bardell, J. C. Bell, C. A. Bennett, S. F. Bibb, W. A. Blankinship, G. A. Bliss, L. M. Blumenthal, D. G. Bourgin, E. L. Buell, H. E. Burns, Herbert Busemann, L. E. Bush, C. C. Camp, W. B. Caton, E. W. Chittenden, R. F. Clippinger, H. B. Curtis, J. Edgar Davis, M. M. Day, John DeCicco, M. W. Dehn, J. M. Dobbie, D. M. Dribin, Samuel Eilenberg, H. S. Everett, L. R. Ford, R. H. Fox, J. H. Giese, J. W. Givens, Michael Golomb, G. D. Gore, Cornelius Gouwens, L. M. Graves, L. W. Griffiths, V. G. Grove, P. R. Halmos, R. W. Hamming, M. L. Hartung, G. E. Hay, C. T. Hazard, M. H. Heins, E. D. Hellinger, Edward Helly, M. R. Hestenes, Carl Holtom, A. S. Householder, H. K. Hughes, H. D. Huskey, G. K. Kalisch, Wilfred Kaplan, William Karush, W. C. Krathwohl, E. P. Lane, R. E. Lane, R. E. Langer, Richard Lence, A. L. Lewis, A. O. Lindstrum, A. T. Lonseth, Eugene Lukacs, Janet MacDonald, G. W. Mackey, Morris Marden, H. J. Miser, C. W. Moran, Ivan Niven, P. B. Norman, J. I. Northam, C. D. Olds, Isaac Opatowski, M. E. Patno, G. B. Price, Tibor Radó, R. B. Rasmusen, Maxwell Reade, W. T. Reid, Haim Reingold, A. E. Ross, Arthur Saastad, R. G. Sanger, Peter Scherk, O. F. G. Schilling, K. C. Schraut, G. E. Schweigert, C. E. Sealander, M. E. Shanks, A. H. Smith, E. R. Smith, W. S. Snyder, Abraham Spitzbart, R. C. Stephens, A. H. Stone, D. M. Stone, E. B. Stouffer, R. L. Swain, H. P. Thielman, C. J. Thorne, M. K. Toft, S. M. Ulam, J. I. Vass, H. S. Wall, J. V. Wehausen, N. W. Wells, M. E. Wescott, M. D. Wetzel, G. W. Whitehead, L. R. Wilcox, K. P. Williams, L. A. Wolf, R. S. Wolfe, Y. K. Wong, J. W. T. Youngs, Daniel Zelinsky.

Sessions for the reading of contributed papers were held on Friday morning with Professor L. M. Blumenthal presiding, Friday afternoon with Professor Reinhold Baer presiding, and Saturday morning with Vice President L. M. Graves and Professor G. A. Bliss presiding. On Friday afternoon prior to the session for contributed papers, Professor Emil Artin of Indiana University gave an invited address on *The axiomatic treatment of number theory and class field theory*. Professor A. A. Albert presided at this lecture.

All sessions were held in the West lecture hall of the Museum of Science and Industry. At the Saturday morning session Professor G. B. Price read a resolution of appreciation to the Museum officials for providing the Society with a place of meeting.

The Council met on Friday, April 23, at 8 P.M. in the Windermere Hotel.

The acting secretary, Professor W. L. Ayres, announced the elec-

tion of the following nine persons to membership in the Society:

Professor Charles Burton Barker, University of New Mexico;
Mr. William Joseph Barr, Naval Research Laboratory, Washington, D. C.;
Mr. Charles Alexander Cole, Jr., U. S. Marine Corps, Naval Research Laboratory, Washington, D. C.;
Mr. Albert Moore Freeman, Providence, R. I.;
Mr. Richard Delph Geckler, Standard Oil Company, Whiting, Ind.;
Dr. Harry Douglas Huskey, Ohio State University;
Mr. Charles Whaley Pflaum, University of Pennsylvania;
Mr. Tracy Whittelsey Simpson, Marchant Calculating Machine Company, Oakland, Calif.;
Mr. Charles Tyrrell West, Lieutenant (j.g.), U. S. Naval Reserve, Briggs Manufacturing Company, Detroit, Mich.

It was announced that the following person was elected as a nominee of the Mathematical Association of America:

Mr. James W. Beach, Iowa State College.

The following appointments by President M. H. Stone were reported: As a committee to consider the advisability of reducing the term of office of the President of the Society from two years to one, Professors G. C. Evans (chairman), G. D. Birkhoff, A. J. Kempner, Solomon Lefschetz, and R. L. Moore; as representative of the Society at the inaugural ceremonies of the Mexican Mathematical Society on April 9, 1943, Professor P. R. Rider.

It was reported that Professor W. T. Martin had accepted an invitation to give an address at the October, 1943, meeting in New York and Professor Antoni Zygmund had accepted an invitation to give an address at the 1943 Summer Meeting.

The Council decided that the meeting scheduled with the American Association for the Advancement of Science in December, 1943, in Cleveland, be canceled, and that arrangements would be made to hold the 1943 Annual Meeting at another time and place.

On request of the editors of the new Quarterly of Applied Mathematics, the Council authorized the President to appoint a liaison officer to consult with the editors of that journal on problems of policy.

Titles and cross references to the abstracts of papers read at the meeting follow below. Papers 1 to 8 were read Friday morning, papers 9 to 14 on Friday afternoon, papers 15 to 26 on Saturday morning, and papers 27 to 37, whose abstract numbers are followed by the letter *t*, were read by title. Paper 1 was read by Dr. Kaplan, 4 by Professor Eilenberg, 12 by Miss Wetzel, and 18 by Dr. Helsel.

1. Wilfred Kaplan and Max Dresden: *Topology of the molecular N-body problem.* (Abstract 49-5-158.)

2. J. W. T. Youngs: *On parametric representations of surfaces.* III. (Abstract 49-5-177.)

3. Tibor Radó: *On continuous path-surfaces of zero area.* (Abstract 49-5-174.)

4. Samuel Eilenberg and Saunders MacLane: *On a group construction by Hopf.* (Abstract 49-1-95.)

5. R. H. Fox: *Natural systems of homomorphisms.* Preliminary report. (Abstract 49-5-172.)

6. A. H. Stone: *Connectedness and coherence.* (Abstract 49-3-124.)

7. G. E. Schweigert: *Fixed elements and periodic types for homeomorphisms on semi-locally-connected continua.* (Abstract 49-5-175.)

8. B. H. Arnold: *On decompositions of T_1 spaces.* (Abstract 49-5-170.)

9. A. E. Ross: *Positive quaternary quadratic forms representing all integers with at most k exceptions.* (Abstract 49-5-129.)

10. A. A. Albert: *Algebras derived by non-associative matrix multiplication.* (Abstract 49-3-103.)

11. Leon Alaoglu: *Harmonic analysis of stochastic processes.* Preliminary report. (Abstract 49-5-169.)

12. H. S. Wall and Marion D. Wetzel: *Contributions to the analytic theory of J-fractions.* Preliminary report. (Abstract 49-5-153.)

13. R. E. Lane: *The values of continued fractions and their computation.* (Abstract 49-5-142.)

14. John DeCicco: *Extensions of certain dynamical theorems of Halphen and Kasner.* (Abstract 49-3-119.)

15. R. W. Hamming: *The asymptotic location of the roots of exponential sums having polynomial exponents.* (Abstract 49-5-136.)

16. S. M. Ulam: *Theory of the operation of product of sets.* II. Preliminary report. (Abstract 49-5-151.)

17. M. M. Day: *Uniform convexity.* III. (Abstract 49-3-111.)

18. R. G. Helsel and Tibor Radó: *The transformation of double integrals.* (Abstract 49-5-137.)

19. G. B. Price: *Cauchy-Stieltjes and Riemann-Stieltjes integrals.* (Abstract 49-1-38.)

20. W. S. Snyder: *A note on the interiority of real functions.* Preliminary report. (Abstract 49-5-147.)

21. P. R. Halmos: *Approximation theories for measure preserving transformations.* (Abstract 49-5-135.)

22. H. D. Huskey: *Contributions to the problem of Geöcze.* (Abstract 49-5-140.)

23. H. K. Hughes: *On a theorem of Newsom.* (Abstract 49-5-139.)

24. Isaac Opatowski: *An explicit formula for the refractive index in*

electron optics. (Abstract 49-5-159.)

25. M. R. Hestenes: *On the condition of Weierstrass in the calculus of variations.* (Abstract 49-7-186.)

26. D. G. Bourgin: *On a quasi-linear operation.* (Abstract 49-5-171.)

27. L. W. Cohen: *On linear equations in Hilbert space.* (Abstract 49-5-131-*t*.)

28. G. E. Albert and L. H. Miller: *Equiconvergence of series of orthonormal polynomials.* Preliminary report. (Abstract 49-3-108-*t*.)

29. George Piranian: *On the convergence of certain partial sums of a Taylor series with gaps.* (Abstract 49-3-112-*t*.)

30. G. E. Albert: *An extension of Korous' inequality for orthonormal polynomials.* Preliminary report. (Abstract 49-3-107-*t*.)

31. Rufus Oldenburger: *The characteristic of a sum of quadratic forms.* (Abstract 49-1-12-*t*.)

32. G. E. Schweigert: *Minimal A-sets, infinite orbits, and fixed elements.* (Abstract 49-5-176-*t*.)

33. W. J. Thron: *Twin convergence regions for continued fractions.* (Abstract 49-5-150-*t*.)

34. J. E. Wilkins: *A special class of surfaces in projective differential geometry.* (Abstract 49-5-166-*t*.)

35. J. F. Harding and Isaac Opatowski: *An approximate formula for the Legendre elliptic integral of the second kind.* (Abstract 49-5-157-*t*.)

36. George Piranian: *On the order of analytic functions in the sense of Hadamard.* (Abstract 49-7-189-*t*.)

37. Reinhold Baer: *A theory of crossed characters.* (Abstract 49-7-178-*t*.)

38. Reinhold Baer: *Radical extensions and crossed characters.* (Abstract 49-7-179-*t*.)

W. L. Ayres,
Associate Secretary

THE APRIL MEETING AT STANFORD UNIVERSITY

The three hundred ninety-seventh meeting of the American Mathematical Society was held at Stanford University on Saturday, April 24, 1943. The attendance was about fifty-five, including the following twenty-nine members of the Society:

H. M. Bacon, K. E. Benson, B. A. Bernstein, H. F. Blichfeldt, T. C. Doyle, S. N. Hallam, M. A. Heaslet, J. G. Herriot, Alfred Horn, D. H. Lehmer, S. H. Levy, J. V. Lewis, A. D. Michal, F. R. Morris, W. H. Myers, George Pólya, R. M. Robinson, J. B. Robinson, Robert Russell, S. A. Schaaf, A. C. Schaeffer, D. C. Spencer, Gabor Szegö, A. E. Taylor, S. P. Timoshenko, J. V. Uspensky, A. R. Williams, František Wolf, Max Zorn.

The meeting opened in the morning with a general session for contributed papers at which Professor Gabor Szegö presided. By invitation of the Program Committee, Professor A. E. Taylor of the University of California at Los Angeles delivered an hour address on *Analysis in complex Banach spaces*. Professor B. A. Bernstein presided at this lecture.

The Symposium on Applied Mathematics was held in the afternoon with Professor A. D. Michal presiding. Professor S. P. Timoshenko of Stanford University spoke on *Theory of suspension bridges*, Dr. E. G. Keller of Lockheed Aircraft Corporation on *Some present nonlinear problems of the electrical and aeronautical industries*, and Dr. Hsu-shen Tsien of the California Institute of Technology on *The "limiting line" in mixed subsonic and supersonic flows of compressible fluids*.

Titles and cross references to the abstracts of the papers read follow below. Papers whose abstract numbers are followed by the letter *t* were read by title. The papers numbered 1 to 4 were read at the morning session; papers 5 and 10 were read in the afternoon; and those numbered 6 to 9 were read by title.

1. R. N. Robinson: *Analytic functions in circular rings.* (Abstract 49-3-113.)

2. B. A. Bernstein: *Postulate sets for Boolean rings.* (Abstract 49-5-167.)

3. George Pólya: *Inequalities for the area of the ellipsoid.* (Abstract 49-5-145.)

4. D. H. Lehmer: *On Ramanujan's numerical function* $\tau(n)$. (Abstract 49-5-128.)

5. Max Zorn: *Informal note on the second underivability theorem* (Abstract 49-5-168.)

6. R. C. James: *Orthogonality in normed linear topological spaces.* (Abstract 49-5-173-*t*.)

7. F. A. Valentine: *A Lipschitz condition preserving extension for a vector function.* (Abstract 49-5-152-*t*.)

8. Glynn Owens: *A boundary value problem for an ordinary nonlinear differential equation of the second order.* (Abstract 49-5-144-*t*.)

9. D. H. Hyers: *A note on the Michal differential in Banach spaces.* (Abstract 49-5-141-*t*.)

10. T. C. Doyle: *Tensor theory of invariants for the projective differential geometry of a curved surface.* (Abstract 49-7-194.)

A. D. MICHAL,
Associate Secretary

BOOK REVIEWS

Topics in topology. By Solomon Lefschetz. (Annals of Mathematics Studies, no. 10.) Princeton University Press, 1942. 139 pp. $2 00.

This book is a companion volume to the author's recently published *Algebraic Topology*. In the latter volume the interest is focused, as the title implies, on the algebraic structure and properties of complexes and on their applications to the study of abstract spaces. Here, however, the interest lies in the topological structure of complexes and related spaces. Indeed, the major part of the book is devoted to a discussion of classes of spaces which are topological generalizations of complexes in much the same way that the abstract complex of *Algebraic Topology* is an algebraic generalization of a finite simplicial complex.

There have appeared in the literature definitions of two such classes which are successful from the point of view of giving rise to interesting and well developed bodies of theorems. The first of these, due to Borsuk, is the class of absolute neighborhood retracts, and the second, due to Lefschetz, is the class of LC* spaces. Both these classes have come to be of considerable topological importance, so the development given here of their essential properties, and of their mutual relations, will be welcomed by all topologists.

There are four chapters. The first is a discussion of various topologies which can be imposed on an infinite complex. A "natural" topology is obtained by taking as a basis for open sets the stars of vertices in all subdivisions. An interesting metric can also be defined and the two topologies are equivalent when the complex is locally finite, the case most used in later applications.

Chapter II is an extension of the discussion in *Algebraic Topology* of singular and continuous cycles. A few new notions along these lines are developed for later use.

In Chapter III a number of topics are dealt with, all dealing with existence theorems for various types of mappings of one space into another. First the Alexandroff theorem on the mapping of a space into the nerve of one of its coverings is generalized by giving necessary and sufficient conditions on the covering for the existence of such a mapping. This has interesting applications to Tychonoff and normal spaces, leading among other things to a characterization of the former.

Next are the proofs of two of the fundamental theorems from di-

mension theory: any n-dimensional separable metric space can be imbedded in a compact metric space of the same dimension and also in the Euclidean space of dimension $2n+1$.

The chapter concludes with a discussion of Borsuk's theory of retraction. An absolute retract (AR) is a separable metric space such that whenever it is a closed subset of a larger space then there is a continuous mapping (a "retraction") of the larger space onto the subset which leaves fixed each point of the subset. An absolute neighborhood retract (ANR) is defined in the same fashion but with the retraction required to exist only on a neighborhood of the subset. The most useful properties of such spaces stem from their characterizations in terms of extensions of mappings, e.g., a necessary and sufficient condition for a space A to be an AR is that whenever B is a closed subset of C, then every mapping of B into A can be extended to a mapping of C into A.

The fundamental theorem proved here, due originally to the author, is that to any separable metric space there may be added an infinite complex in such a fashion that the resulting union is an absolute retract.

The final chapter is devoted to the author's theory of homotopy local connectedness. Roughly speaking, a space is p-LC if small singular p-spheres in the space can be shrunk to points over small sets. The interesting case is where the space is LC^n, i.e., p-LC for all $p \leqq n$. Such spaces can be characterized in terms of mappings of complexes into the space, and this leads to the slightly stronger notion of an LC* space. The fundamental theorem is that a compact absolute neighborhood retract is an LC* space and conversely.

Next the homology theory of such spaces is investigated. It turns out that all types of cycles (e.g., singular, Vietoris, Čech) yield the same homology groups. Furthermore these groups can be no more complicated than those of a finite complex; in particular they have finite bases. Using this result, the fixed point formula is shown to hold for these spaces. As an application the Schauder fixed point theorem for functional spaces and some of its generalizations are derived.

The chapter concludes with a brief outline of the results obtained by generalizing the local connectedness definitions by replacing homotopy by homology, i.e., singular spheres by cycles and shrinking to a point by bounding.

As with all the volumes of this series, this one is well printed and easy to read. It is marred, however, by an unfortunate number of misprints, although few of them will cause the reader any particular difficulty. There are also a few slips. The proof of the last part of the

last theorem of Chapter III is incorrect but is easy to remedy. In theorem 23.1 of Chapter IV, the inequality should read "$q \leqq p$." The proof here is not quite complete.

The terminology and notations used here are those of *Algebraic Topology*, so the index and bibliography are brief. There is, however, an additional bibliography of papers dealing with the subjects of retraction and homotopy local connectedness which is quite complete and useful.

EDWARD G. BEGLE

The calculus of extension. By Henry George Forder. (Including examples by Robert William Genese.) Cambridge University Press; New York, Macmillan, 1941. 15+490 pp. $6.75.

It will soon be the hundredth anniversary of the initial publication of Grassmann's monumental work *Die Lineale Ausdehnungslehre Ein Neuer Zweig der Mathematik*. Perhaps this fact will serve to arouse some belated interest in Grassmann's celebrated but otherwise neglected contributions to mathematics. If so, the excellent volume under review should indeed be a welcome addition to the comparatively meagre supply of general works treating this subject.

The history of the *Ausdehnungslehre* is extremely depressing. Grassmann had hoped that he would be able to secure the special opportunities for mathematical research that naturally accrue to a university post. In this he was bitterly disappointed. Such a position was not forthcoming. Cajori writes[1] "At the age of fifty-three this wonderful man, with heavy heart, gave up mathematics, and directed his energies to the study of Sanskrit, achieving in philology results which were better appreciated, and which vie in splendor with those in mathematics." Grassmann presented the *Ausdehnungslehre* in two books. The first, *Die Lineale Ausdehnungslehre*, appeared in 1844 (second edition, 1878) and the second, *Die Ausdehnungslehre, vollständig und in strenger Form bearbeitet*, in 1862. It has been said that only one person had read through the *Ausdehnungslehre* of 1844 eight years after its publication. Apparently, it was too new, general, and abstract to meet with popular approval. The *Ausdehnungslehre* of 1862 is easier to read and gives ample evidence of the wide range of the applications. It fared but little better than the former. In recent times Grassmann's work has been better appreciated. The historians of our subject[2] com-

[1] See Florian Cajori, *A history of mathematics*, New York, 1938.

[2] See, E. T. Bell, *The development of mathematics*, New York, 1940; Florian Cajori, *A history of mathematics*, New York, 1938; J. L. Coolidge, *A history of geometrical methods*, Oxford, 1940.

mend it, but it has not yet been adopted in America to the extent that it is frequently presented in courses and employed in exposition and research. This is in strong contrast to vector analysis which was "discovered" shortly after its extraction, by Gibbs and others from the *Ausdehnungslehre* and Hamilton's *Quaternions*. Tensor analysis, a second subject which had roots in the *Ausdehnungslehre*, was rather generally discovered twenty years after its formulation by Ricci and Levi-Civita.

Since the *Ausdehnungslehre* is not yet widely known, one or two remarks on its nature are in order. First, as to its formal aspects, this subject may be described as a study of the linear form $\sum x_i e_i$, in which the x's are numbers and the e's (the extensives) are mathematical entities following a prescribed algebra. In the geometrical applications, the extensives are geometrical entities, such as points, vectors, rotors (a rotor is a vector confined to a line through it), circles, etc. The definitions of the sums, differences, and products of the extensives form the means whereby geometrical theorems are reduced to algebraic identities. While the machinery contains numerous minutiae (definitions, rules of operation, etc.) the separate details are quite simple and the resulting theory is extremely powerful. In a word, the general subject matter of *The calculus of extension* is of considerable interest both historically and intrinsically.

With regard to the book itself, the important questions are the following: Does the author present new material and if so is the new material developed with clarity? Is old material rendered more easily attainable for the uninitiated? For what purposes is the book suited? Forder's *The calculus of extension* not only presents a wealth of specific applications of the subject to geometry, that are either new or not readily obtainable elsewhere; but in addition furnishes an admirable and fresh exposition of the *Ausdehnungslehre*. It is very carefully written and the individual proofs are uniformly short and easy to follow. The theorem density is exceptionally high and consequently despite the superior exposition it is not an easy book to work straight through—perhaps the key chapters suffer from lack of recapitulation. Books that contain a wealth of material are never easy to read through, and it is my conviction that *The calculus of extension* provides the best exposition of the fundamental processes of the *Ausdehnungslehre* and the most inclusive treatment of the geometrical applications available at present. It is a book that should be in the library of anyone who is interested in either algebra, the algebraic treatment of geometry, or vector and tensor analysis. Certain chapters could be used effectively as the basis for a college course.

Turning to the negative matter of the accuracy of the work, there is little that needs to be said. It is not feasible, of course, to produce a book without a blemish and I have noticed a few misprints or slips of the pen and one or two places that would benefit by revision. These flaws, however, are remarkably scarce and are not of the type that will cause the reader difficulty, assuming that he is ready for the subject. Consequently, they have but slight bearing on the purpose of this review which is, I presume, to enable the reader to decide whether or not he wishes to examine the book. I should have preferred a more modern treatment of differentiation, but this is not a serious matter since the book is obviously addressed to readers who have the maturity to supply the rigorous formulation. The statement on page 221 that a combination not satisfying $a(b+c)d=abd+acd$ would not be called a product seems a little strong in the light of the dot product of vector analysis; and number 13 on the preceding page needs revision. These are small matters, however, and as I have said the book is unusually accurate.

For the uninitiated who wish to get a general notion of the subject as quickly as possible a careful study of Chapter I, together with a sampling of Chapters II, V, VII, VIII, and IX will suffice. Chapter I (71 pages) begins with a polished treatment of the fundamentals of the algebra of extensives and then proceeds with the interpretation of the primary extensives as points in a plane. The product of a point and a real number k is a point of weight k. The difference $b-a$ of two points of unit weight is the vector $\vec{ab}$. The sum of a point and a vector is the terminus of the vector when the vector's origin is placed at the point. The sum of two weighted points is a point at their center of gravity with weight equal to the total weights of the original points. The outer product of two points a and b is the rotor obtained by confining the vector $b-a$ to the line through a and b. The outer product of three points $[abc]$ is twice the area of the triangle (a, b, c). The difference of two rotors whose vectors are equal is called a bivector and in accordance with the equation $[(a-b)(p-q)]=[a(p-q)]-[b(p-q)]$ the outer product of two vectors is taken to be a bivector. The inner product of two vectors u and v is defined in terms of the supplement of v, i.e., the vector obtained by rotating v through 90 degrees in the positive direction. Specifically, the inner product of u and v is the outer product of u and the supplement of v. Since a bivector may be identified with its magnitude, namely, the area of an associated parallelogram, the inner product of two vectors is essentially the dot product of vector analysis. On the basis of these and certain other definitions the author establishes a wide variety of geometrical

theorems including some in projective geometry.

Chapter II (pp. 71–110) is similar in character to Chapter I except that it is concerned with three-space. Chapter III (pp. 111–155), together with Chapters I, II, and VI, gives a reasonably complete account of projective geometry. Chapter V, Differentiation and Motion (pp. 177–196) is somewhat similar to the development by classical vector analysis. It treats among other things velocity, acceleration, curvature, central motion, and displacements in space. In Chapter VII, The General Theory (pp. 217–249), the central ideas are linear dependence and the various products of extensives. Determinants enter the picture by way of the fact that if $b_1, \cdots, b_r$ are each linearly related to $a_1, \cdots, a_r$, then $[b_1 \cdots b_r] = D[a_1 \cdots a_r]$, D being the determinant of the coefficients in the linear relationship. Chapter VIII (pp. 250–264) gives the application to linear equations and determinants. With regard to linear equations the modus operandi is to multiply each equation of the set $a_\rho^\tau x^\rho = b^\tau$ by an extensive e_τ and add, thus $e_\tau a_\rho^\tau x^\rho = e_\tau b^\tau$. This reduces N equations in N unknowns to a single equation in extensives. The familiar theorems on the solvability of equations follow with marked simplicity. The treatment of determinants like the book as a whole is characterized by the wide scope of the material which is intelligibly presented in a relatively small space. Chapter IX (pp. 265–294) treats transformations, square matrices, and central quadrics—mostly well known material in slightly different dress. The remaining chapters are devoted to: the screw and linear complex, the general theory of inner products, circles (two chapters), the general theory of matrices, quadric spreads, and algebraic products.

HOMER V. CRAIG

Introduction to the theory of relativity. By Peter Gabriel Bergmann with a foreword by Albert Einstein. New York, Prentice-Hall, 1942. 287 pp. $4.50.

Many excellent books have been written on the Theory of Relativity. Although some of them appeared more than twenty years ago they are still read and studied, far from being regarded as antiquated. The books of Weyl, Pauli, Eddington are justly looked upon as classics in this subject. To say that Bergmann's book is in the same class as the books just mentioned means great, but deserved, praise.

Bergmann's book has its own character, and differs from the other books on Relativity Theory which have appeared up to now. First, it is more modern. The application of Relativity Theory to the Theory

of Cosmic Rays, to Sommerfeld's relativitistic theory of the hydrogen atom, de Broglie's waves, Ives' experiments, Compton's effect, all these topics have found their proper places in Bergmann's book. But the advantage of having all these subjects included in a book on Relativity Theory does not, in itself, mean very much. Relativity Theory had changed but little in the last twenty-five years and all these additions are not very important from the point of view of Relativity Theory. Much more important for the evaluation of this book is the originality of the author's approach.

Someone may look for a book on Relativity Theory which states clearly and axiomatically the assumptions of this theory and develops deductively the conclusions from these assumptions. This is not what Bergmann's book tries to do. What it tries to do, and does excellently, is to show how we were compelled to adopt these assumptions, how the structure of Relativity grew from logical contradictions in the classical theory, how their removal leads naturally and simply to the Theory of Relativity. The author presents not the painful historical process, not how Relativity was discovered, but how it should have been discovered if we had known the simple and straight road of logic leading to its formulation. Even in Relativity Theory, created almost by the genius of one man, this difference between the historically and logically reconstructed process is remarkable; it is the difference between the broad highway and the pioneer's narrow pathway.

The author succeeds well in presenting the development of Relativity as a logical necessity. The book is always free from the dogmatic dull textbook tone and often achieves dramatic qualities. The style is clear and simple.

The book is divided into three parts. The first part (pp. 3–147) contains the Special Relativity Theory. Tensor algebra is included in this part. Tensor analysis is partially in this part, partially in the second part. Though from the structure of the book it is clear why the author did it, I have my doubts whether this was the best possible solution. Compared with the material of other books, one finds in this part modern examples from Quantum Theory, but misses the Minkowski representation.

The second part (pp. 151–242) contains the General Relativity Theory. The transition from the Special to General Relativity Theory is remarkably good. The whole chapter shows the natural development of the essential ideas in General Relativity Theory; difficult concepts are introduced in a very interesting and simple, though never superficial manner. When comparing it with other treatments,

we note the omission of cosmological applications (treated so well in Tolman's book) and of time-space measurements in connection with the experimental verifications of General Relativity Theory. On the other hand much space is devoted to the equations of motion as deduced from the field equations.

The third part (pp. 245–279) is of much more special character and deals with the unification of the gravitational and electromagnetic field. Here we find an exposition of Weyl's and Kaluza's theories and of their generalizations on which the author collaborated with Einstein. This part will rather interest specialists than students.

The book is well designed. The title of the book is too modest. It is not an introduction; it is an excellent book on the principles of Relativity Theory.

L. INFELD

Degree of approximation by polynomials in the complex domain. By W. E. Sewell. (Annals of Mathematics Studies, no. 9.) Princeton University Press, 1942. 9+236 pp. $3.00.

The main subject of this book is the relation of the analytic and continuity properties of a function inside and on the boundary of a region, and the degree of approximation by polynomials. It is related to the analogous questions for functions of a real variable and approximation by trigonometric sums, as presented in treatises of C. de la Vallée Poussin, S. Bernstein, D. Jackson and G. Szegö. The problems of the present work originate with the fundamental theorem that a function analytic in a Jordan region and continuous in the closed region can be uniformly approximated by polynomials. This was proved by J. Walsh in 1926. His treatise *Interpolation and approximation by rational functions in the complex domain* (Amer. Math. Soc. Colloquium Publications, vol. 20, 1935) is an important forerunner to the present book.

This book is divided into two parts. Part 1 (Chapters II, III, and IV) is devoted to a study of the relation between the degree of convergence of certain sequences of polynomials $p_n(z)$ to a function $f(z)$ on a point set E in the z-plane on the one hand and the continuity properties of $f(z)$ on the boundary C of E on the other hand (Problem α). Recent contributions to Problem α are due primarily to J. Curtiss, Sewell, and Walsh.

In Part II (Chapters V–VIII) a more delicate problem is considered. Let E with boundary C be a closed limited set, whose complement K is connected and regular; thus a function $w=g(z)$ maps K conformally on the exterior of the unit circle $|w|=1$ in the w-plane so

that the points at infinity in the two planes correspond to each other. Denote by C_ρ the image, under the conformal map, in the z-plane of the circle $|w| = \rho > 1$. Let $f(z)$ be given analytic interior to some fixed C_ρ and let its continuity properties on C_ρ be prescribed. Problem β is the study of the relation between the degree of convergence of various sequences $p_n(z)$ to $f(z)$ on E on the one hand and the continuity properties of $f(z)$ on C_ρ on the other hand. Special cases of this problem have been studied by Bernstein, G. Faber, and de la Vallée Poussin. The present work discusses in detail more recent material primarily due to Sewell and Walsh.

Two types of results are considered in either Problem α or β. A direct theorem is one in which the properties of $f(z)$ are in the hypothesis and the degree of convergence of $p_n(z)$ to $f(z)$ is the conclusion. An indirect (generally far more difficult) theorem is in the converse direction, that is the degree of convergence of $p_n(z)$ to $f(z)$ is in the hypothesis and the properties of $f(z)$ form the conclusion. Such properties are: (a) A Lipschitz condition of order α is satisfied by $f(z)$ on a set E; that is $|f(z_1) - f(z_2)| \leqq L|z_1 - z_2|^\alpha$, where α is a fixed number, $0 < \alpha \leqq 1$, z_1 and z_2 are arbitrary points of E, and L is a constant independent of z_1 and z_2. (b) $f(z)$ belongs to the class $L(\kappa, \alpha)$ on a Jordan curve C (boundary of E), if $f(z)$ is analytic in the interior points of E, is continuous in the closed set E, and the κth derivative $f^{(\kappa)}(z)$ exists on C in the one-dimensional sense and satisfies a Lipschitz condition of order α on C. (c) $f(z)$ belongs to the class $\log(\kappa, 1)$ on C if $f(z)$ is analytic in the interior points of E, is continuous on the closed set E, and $f^{(\kappa)}(z)$ exists on C in the one-dimensional sense and satisfies the condition

$$|f^{(\kappa)}(z_1) - f^{(\kappa)}(z_2)| \leqq L|z_1 - z_2||\log|z_1 - z_2||,$$

where z_1, z_2 are arbitrary points of C, $|z_1 - z_2|$ sufficiently small ($< 1/2$ say); L is a constant.

Occasionally an integrated Lipschitz condition is also considered. The degree of convergence or approximation is measured:

1. In the sense of Tchebycheff, that is by

$$\text{l. u. b.}_{z \text{ in } E} |f(z) - p_n(z)|.$$

2. By a line integral:

$$\int_C \Delta(z) |f(z) - p_n(z)|^q |dz|,$$

where $\Delta(z)$ is a non-negative weight function, q is a fixed positive number.

As the author points out, many of the results are not final; at the end of each chapter possible extensions are indicated. The bibliography is restricted to a selected list. A few scattered misprints and errors can be easily corrected. For example on p. 13 the inequality $|\partial P/\partial\theta| \leqq A/(1-r)$ is false, but the proof of the theorem can be completed. On p. 126, where a beautiful theorem of Hardy and Littlewood is quoted, reference should be made to a paper of E. S. Quade, Duke Math. J., vol. 3 (1937), pp. 529–543.

On the whole the author deserves credit for his valuable contribution which will serve to stimulate further research on this important subject.

OTTO SZÁSZ

Table of arc-tan x. Federal Works Agency, Works Projects Administration for the City of New York, 1942. 25+173 pages. $2.00.

"This table of arc-tan x is believed to be the most comprehensive so far published, in respect both to the number of decimal places given and to the smallness of tabular interval. It forms the first contribution to what it is hoped will be a complete set of tables of the inverse trigonometric and hyperbolic functions." The tables are computed to twelve decimals. The interval between successive arguments is 0.001 for $0 \leqq x \leqq 7$, it is 0.01 for $7 \leqq x \leqq 50$, 0.1 for $50 \leqq x \leqq 300$, 1 for $300 \leqq x \leqq 2{,}000$ and 10 for $2{,}000 \leqq x \leqq 10{,}000$. Linear interpolation provides for the whole range an accuracy of six decimal places. For interpolation to twelve places the second central differences are tabulated. The introduction gives the necessary formulae; correspondingly tables to six places for $p(1-p)$ and $1/6\ p(1-p^2)$ are given for the range $0 \leqq p \leqq 0.5$ and $0 \leqq p \leqq 0.999$. Tables for the conversion of radians into degrees and conversely are added. The introduction gives the necessary information for the use and scope of the tables, and ends in a report about the method of checking by a sixth difference test. The bibliography contains also a new list of errors in the tables of Hayashi.

O. NEUGEBAUER

NOTES

The American Mathematical Society announces a new series of monographs, Mathematical Surveys. Each book will be about 150 pages in length and will make available the principal methods and results that have been obtained on a certain broad problem, without entering into all details of proof. The books of the new series will be distinguished from the Colloquium Publications in that the latter contain, to a large extent, new research material, while the monographs in the surveys will be mainly expository.

The first two books of the series are the following: *The problem of moments* by J. D. Tamarkin and J. A. Shohat and *Rings* by Nathan Jacobson. The price for Volume 1 is $2.50 and for Volume 2 $2.00. Members of the American Mathematical Society receive a discount of twenty-five per cent. Orders for either of these books should be addressed to the Society at 531 West 116th Street, New York, 27, N. Y.

The First National Congress of Mathematics was held at Saltillo, Mexico, November 1–7, 1942. Dr. A. Nápoles Gándara, who is director of the Institute of Mathematics at the University of Mexico, was secretary of the Congress. Professor P. R. Rider of Washington University and Professor M. S. Vallarta of the Massachusetts Institute of Technology participated in the Congress.

Dr. P. A. Alexandroff has been awarded a Stalin prize.

Professor L. E. J. Brouwer of the University of Amsterdam and Professor Godofredo García Díaz of the University of San Marcos have been elected to membership in the American Philosophical Society.

Professor E. A. Milne of the University of Oxford has been elected president of the Royal Astronomical Society.

Professor G. F. J. Temple of Kings College, London, has been elected to fellowship in the Royal Society.

Professor A. A. Albert of the University of Chicago has been elected to membership in the National Academy of Sciences.

Professor G. D. Birkhoff of Harvard University has been elected an honorary member of the Royal Irish Academy. He has also received an honorary doctorate of laws from the University of California at Los Angeles.

Professor M. H. Stone of Harvard University has been elected to membership in the American Philosophical Society.

Professor J. L. Synge of the University of Toronto has been elected a fellow of the Royal Society of London.

Dr. P. C. Rosenbloom has been appointed a National Research Fellow.

Assistant Professor R. G. Archibald of Queens College has been promoted to an associate professorship.

Assistant Professor H. R. Cooley of New York University has been promoted to an associate professorship.

Assistant Professor Nelson Dunford of Yale University has been promoted to an associate professorship.

Associate Professor J. S. Frame of Allegheny College has been appointed to a professorship at Michigan State College.

Associate Professor W. L. Hutchings of Rollins College, Winter Park, Florida, has been appointed to an associate professorship at Whitman College, Walla Walla, Washington.

Mrs. C. J. Keyser has been appointed dean of Nightingale-Bamford School, New York City.

Dr. B. A. Lengyel of the College of the City of New York has been appointed to an assistant professorship in the department of physics at the University of Rochester.

Assistant Professor W. T. Martin of Massachusetts Institute of Technology has been appointed to a professorship at Syracuse University.

Associate Professor A. F. Moursund of the University of Oregon has been promoted to a professorship.

Professor J. L. Synge of the University of Toronto has been appointed head of the department of mathematics at Ohio State University to replace Professor H. W. Kuhn, who has retired.

Dr. A. D. Wallace of the University of Pennsylvania has been promoted to an assistant professorship.

Associate Professor V. H. Wells of Williams College has been promoted to a professorship.

Dr. Max Wyman has been appointed lecturer in mathematics at the University of Alberta.

Dr. Leo Zippin of Queens College has been promoted to an assistant professorship.

The following appointments to instructorships are announced: University of Buffalo: Dr. Paul Civin; University of Michigan: Mr. P. S. Jones; Reed College; Dr. R. W. Shephard; University of Virginia: Mr. W. H. Gottschalk.

Dean H. E. Hawkes of Columbia University died May 4, 1943, at the age of seventy years. He had been a member of the Society since 1898.

Professor R. L. Menuet of Tulane University died May 9, 1943.

Dr. W. H. Metzler who retired as dean of the New York State College for Teachers in 1933 died April 18, 1943, at the age of seventy-nine years. He had been a member of the Society since 1891.

Professor Emeritus H. S. White of Vassar College died May 20, 1943. He had been a member of the Society since 1892.

Professor J. E. Williams of Virginia Polytechnic Institute died April 19, 1943, at the age of seventy-five years. He had been a member of the Society since 1905.

Professor Emeritus Clara E. Smith of Wellesley College died May 12, 1943. She had been a member of the Society since 1904.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

ALGEBRA

178. Reinhold Baer: *A theory of crossed characters.*

If G is a finite group, C a homomorphism of G into the group of automorphisms of the cyclic group E, then a C-character of G is any single valued G to E function $f(x)$ satisfying the functional equation: $f(uv)=f(u)^{v}f(v)$ for u, v in G where the exponent v indicates the automorphism upon which the element v is mapped by C. The C-characters of G form a finite abelian group; and it is the object of a theory of crossed characters to find conditions assuring the possibility of establishing a partial or complete duality between G and its C-character group. (Received April 23, 1943.)

179. Reinhold Baer: *Radical extensions and crossed characters.*

An m-extension of the field F is a finite, normal and separable extension of F which is obtained by adjoining to F mth roots of elements in F. A theory of these extensions may be obtained by applying the theory of crossed characters on the Galois group of these extensions. (Received April 23, 1943.)

180. C. J. Everett: *Closure operators and galois theory in lattices.*

Every *-closure ($A^{*}\supset A$; $A\subset B$ implies $A^{*}\subset B^{*}$) on a partially ordered set arises from a galois correspondence between P and some partially ordered set Q. Every galois correspondence between complete lattices of subsets of two sets results from a binary relation on all elements of the two sets. Every closure on a complete lattice P of subsets of a set is extensible to all subsets; if $(A\cup B)^{*}=A^{*}\cup B^{*}$ in P, the extension is also topological, for P a boolean algebra. Every closure on all subsets of a set is defined by a binary relation. A generalization of Krull's topology for algebraic fields is obtained and used to characterize the regularly closed subspaces of the linear functional space of a Banach space. Necessary and sufficient conditions are given for the existence of a topology in a group on whose subgroup lattice a *-closure is defined, such that a subgroup is *-closed if and only if it is closed in the topology. This is combined with results of Baer on primary groups to characterize topologically the subgroups closed under the galois correspondence between P and its automorphism group, A. Krull's method converts A into a metric, totally disconnected, topological group. (Received May 29, 1943.)

181. Irving Kaplansky: *Solution of the "problème des ménages."*

The "problème des ménages" asks for the number of ways of seating n husbands and n wives at a circular table, men alternating with women, so that no husband

sits next to his wife. By elementary methods, the author obtains the answer $4n\,n!\sum_{k=1}^{n}(-1)^k C_k^{2n-k}(n-k)!/(2n-k)$. (Received June 1, 1943.)

ANALYSIS

182. E. F. Beckenbach and R. H. Bing: *Conformal minimal varieties.*

A set of m real functions of n real variables, $m \geqq n \geqq 2$, defined in a domain D, has been called (for $m=n$ by N. Cioranesco, Bull. Sci. Math. vol. 56 (1932) pp. 55–64) a set of conjugate harmonic functions provided the functions are harmonic and together satisfy the usual conditions for conformality. Such a set of m functions gives a conformal map on Euclidean n-space of a minimal variety V_n immersed in Euclidean m-space. But according to Haantjes, for $n \geqq 4$ the class of conformally flat spaces is quite narrow. It is now shown directly that for $n \geqq 3$ the only sets of conjugate harmonic functions necessarily are constants or linear functions, so that either the V_n is a point or it can be obtained from D by rigid motions, transformations of similitude, and reflections in hyperplanes. (Received May 7, 1943.)

183. R. P. Boas: *Almost periodic functions of exponential type.*

As an extension of a well known result on entire functions of exponential type which are periodic on the real axis, it is shown that an entire function of exponential type which is almost periodic on the real axis has its Fourier exponents bounded. Almost periodicity as general as Besicovitch (order 1) is admissible; in fact, the mere existence of the mean value of $|f(x)|$ implies the existence and vanishing of the mean value of $f(x)e^{i\lambda x}$ when $|\lambda|$ exceeds the type of the function $f(z)$. For the proof, one rotates the line of integration in $\int_0^T e^{i\lambda z} f(x)dx$ through an angle of $\pi/2$, and applies theorems which connect the growth of $|f(x+iy)|$ with that of $|f(x)|$. The estimate $f(x)=o(|x|)$ as $|x| \to \infty$ is a consequence of the existence of the mean value of $|f(x)|$. (Received May 11, 1943.)

184. Vincent Cowling and Walter Leighton: *On convergence regions for continued fractions.*

Let k be any real number greater than 1 and ϵ any number such that $0<\epsilon<k$. If the complex numbers a_{2n} and $a_{2n+1}=r_n e^{i\theta_n}$ satisfy the conditions (1) $|a_{2n}| \leqq k-\epsilon$, $a_{2n} \neq -1$, (2) $r_n \geqq 2[k-\cos\theta_n]$, $0 \leqq \theta_n \leqq 2\pi$, then the continued fraction $1+K[a_n/1]$ converges. It follows as a corollary that the continued fraction will converge if conditions (1) and $|a_{2n+1}| \geqq 2(k+1)$ hold. (Received April 28, 1943.)

185. Nelson Dunford: *Spectral theory.* I. *Convergence to projections.*

By a systematic use of an operational calculus suggested by the formula $f(T)=(2\pi i)^{-1}\int_C f(\lambda)(\lambda I-T)^{-1}d\lambda$ conditions are derived which are necessary and sufficient for the convergence of a given sequence $P_n(T)$ of polynomials in a linear operator T (on a complex Banach space X) to a projection on a manifold of the form $M[P]=E_0[x \in X, P(T)x=0]$. (Received April 24, 1943.)

186. M. R. Hestenes: *On the condition of Weierstrass in the calculus of variations.*

The present paper is devoted to the study of properties of the Weierstrass E-func-

tion $E(y, p, \lambda, q)$ associated with the problem of Bolza. It is shown that an extremal C_0 is nonsingular and satisfies the strengthened condition of Weierstrass if and only if there is a constant $b>0$ such that the inequality $E(y, p, \lambda, q) \geqq bE_L(p, q)$ holds for all differentiably admissible elements (y, p), (y, q) with (y, p, λ) in a neighborhood of the values (y, y', λ) on C_0. Here E_L is the E-function for the length integral. Further equivalence relations are given. Moreover, it is shown that if these conditions are satisfied then the integrand $f(y, y')$ can be replaced by a function of form $f+\theta(y, y')\phi^\beta\phi^\beta$ such that the above inequality holds without the restriction that $\phi^\beta(y, p)=\phi^\beta(y, q)=0$. (Received April 23, 1943.)

187. Herman Kober: *On the approximation to integrable functions by integral functions.*

Let $f(t)\in L_p(-\infty, \infty)$, $0<p\leqq\infty$, and suppose that $f(t)$ may be approximated (in mean if $0<p<\infty$, uniformly if $p=\infty$) by integral functions of fixed order not greater than ρ, where ρ is fixed. If $\rho<1$, $f(t)$ is a constant (zero if $p=\infty$). Here 1 is the precise limit and it is always possible to find a sequence of integral functions of order one and normal type approximating $f(t)$ (if $p=\infty$, if and only if $f(t)$ is uniformly continuous in $(-\infty, \infty)$). If the types of the approximating functions have a finite inferior limit α, then $f(t)$ is an integral function of order one and type not greater than α, or a constant. The best approximation by integral functions of order one and given type α can be determined and is given by the Dirichlet singular integral $D_\alpha[f]$ when $p=2$. Approximation by functions of order ρ implies approximation by functions of larger order. If $L_p(-\infty, \infty)$ be replaced by $L_p(0, \infty)$, the order one is to be replaced by the order 1/2 throughout. Extensions to other approximation problems are possible. (Received April 17, 1943.)

188. E. R. Lorch: *The theory of analytic functions in normed abelian vector rings.*

The development of an analytic function theory for these rings $\mathfrak{R}$ has been announced previously (Bull. Amer. Math. Soc., abstract 46-11-469). Certain applications are here considered. The function $\exp(z)$ is simply periodic if and only if $\mathfrak{R}$ is irreducible; alternatively, if and only if the spectrum of each $a\in\mathfrak{R}$ is a continuum or a point. The periods of $\exp(z)$ are $2\pi i\sum_1^* \pm j_k$ where the j_k are any idempotents. The function $\log z$ exists for all z in the principal component of the topological group of non-singular elements in $\mathfrak{R}$. An $a\in\mathfrak{R}$ may be embedded in a continuous group, $a^s\cdot a^t=a^{s+t}$, if and only if $\log a$ exists; then $a^s=\exp(s\log a)$ where the range of s may be taken to be $\mathfrak{R}$. All points in the frontier of the spectrum of $a\in\mathfrak{R}$ are permanent singularities (cannot be removed by ring extensions). If $\mathfrak{R}$ is irreducible, a simple closed rectifiable curve C in $\mathfrak{R}$ generates a closed set of singular elements, an open set of exterior elements, and an open set of interior elements. The open sets need not be connected. In case C has a tangent at one point, the set of interior points is not empty. (Received April 19, 1943.)

189. George Piranian: *On the order of analytic functions in the sense of Hadamard.*

If in the region $|z|>R$ the function $f(z)=\sum a_n/z^n$ is holomorphic except for poles of aggregate multiplicity P, if D_{nP} is Hadamard's determinant of order $P+1$ and rank n for $f(z)$, and if the order of the function $F_P(z)=\sum D_{nP}/z^n$ on its circle of convergence is ω, then the order of $f(z)$ on the circle $|z|=R$ is at least ω. Moreover, if the

order of $f(z)$ on $|z|=R$ is $\omega'>\omega$, there either exist at least two points of order ω' on this circle or the singularity of order ω' is non-Fuchsian. By means of an extension of Mandelbrojt's method for finding the singularities of an analytic function on its circle of convergence, the theorem gives a formula for the order of every pole lying outside of the convex hull of non-polar singularities of $f(z)$, and for the order of every Fuchsian singularity on the boundary V of the convex hull, provided the singularity is not an interior point of a straight-line segment of V. (Received April 12, 1943.)

190. Harry Pollard: *A new criterion for completely monotonic functions.*

The Bernstein criterion for completely monotonic functions states that if (i) $f(0+)$ exists and (ii) $(-1)^k\Delta_\delta^k f(x)\geqq 0$ for $k\geqq 0$, $\delta>0$, $x>0$, then $f(x)$ is completely monotonic in $0\leqq x<\infty$. It is established in this paper that (ii) can be weakened to $(-1)^k\Delta_{\delta_k}^k f(x)\geqq 0$ for a suitable sequence $\{\delta_k\}$. (Received April 10, 1943.)

191. W. J. Thron: *A general theorem on convergence regions for continued fractions $b_0+K(1/b_n)$.*

Let the regions B_0 and B_1 be defined by: $r\cdot e^{i\theta}\in B_0$ if $r>(1+\epsilon)\cdot f(\theta)$, $r\cdot e^{i\theta}\in B_1$ if $r>(1+\epsilon)g(\theta)$, where ϵ is an arbitrary small positive number and the functions $f(\theta)$ and $g(\theta)$ are positive in the interval $[0, 2\pi]$. If it is required that the complements of the regions B_0 and B_1 be both convex and if $f(\theta)\cdot g(\pi-\theta)\geqq 4$, then the continued fraction $b_0+K(1/b_n)$ converges if $b_{2n}\in B_0$ and $b_{2n+1}\in B_1$, that is B_0 and B_1 are twin convergence regions for the continued fraction. The condition $f(\theta)g(\pi-\theta)\geqq 4$ is a necessary condition for two regions to be twin convergence regions. (Received April 19, 1943.)

192. W. J. Thron: *Convergence regions for the general continued fraction.*

It is shown that the continued fraction $K(a_n/b_n)$ converges if all $a_n=r\cdot e^{i\theta}$ lie in a bounded part of the parabola $r\leqq a^2/2(1-\cos(\theta-2\gamma))$, and if all b_n lie in the half-plane $R(b_ne^{i\gamma})\geqq a+\epsilon$. Here $a>0$ and ϵ is an arbitrary small positive number. (Received April 24, 1943.)

193. Hassler Whitney: *On the extension of differentiable functions.*

Let A be a bounded closed set in Euclidean space E. Suppose that for some number ω any two points of A are joined by an arc in A of length not more than ω times their distance apart. Then any function of class C^m in A which, with derivatives through the mth order, is sufficiently small in A, may be extended throughout E so as to be small, with its derivatives. (Received May 11, 1943.)

GEOMETRY

194. T. C. Doyle: *Tensor theory of invariants for the projective differential geometry of a curved surface.*

This paper completes the explicit determination of differential invariants of all orders for a curved two dimensional surface begun by E. J. Wilczynski, *Projective differential geometry of curved surfaces (fourth memoir)*, Trans. Amer. Math. Soc. vol. 10 (1909) pp. 176–200. The Lie theory of groups serves to determine the number of exist-

ing invariants of any specified order and these are then exhibited by contraction of fundamental surface tensors and their covariant derivatives appearing as coefficients of a system of partial differential equations defining the surface. The projective normal is attained by the formal elimination procedure of tensor analysis. (Received April 10, 1943.)

195. H. T. Muhly: *Independent integral bases and a characterization of regular surfaces.*

Let $\mathfrak{o}^*$ denote the ring of homogeneous coordinates, $\xi_0^*, \xi_1^*, \cdots, \xi_n^*$ associated with a normal, nonsingular model F, of a field Σ of algebraic functions of two variables, and assume that $\xi_0^*, \xi_1^*, \xi_2^*$ are selected so that they are algebraically independent and so that each element of $\mathfrak{o}^*$ depends integrally upon these three elements. Let the relative degree $[K(\xi_0^*, \xi_1^*, \cdots, \xi_n^*) : K(\xi_0^*, \xi_1^*, \xi_2^*)]$ be denoted by ν. It is shown in this paper that if there exists a set of ν elements $\lambda_1^*, \lambda_2^*, \cdots, \lambda_\nu^*$ in $\mathfrak{o}^*$ which form an independent modular base for $\mathfrak{o}^*$ over the ring $K[\xi_0^*, \xi_1^*, \xi_2^*]$, then the field Σ is regular, that is, its arithmetic genus p_a coincides with its geometric genus p_g. Furthermore, it is shown that if the field Σ is regular then there exist projective models of Σ which are normal and nonsingular, and which are such that the associated ring of homogeneous coordinates has a linearly independent modular base over a suitably chosen ring of independent variables. In fact if F is any normal, nonsingular model of a regular field Σ, and if F_h is the derived normal model belonging to the character of homogeneity h, then F_h has these properties for all sufficiently large values of h. (Received May 15, 1943.)

Numerical Computation

196. A. N. Lowan and H. E. Salzer: *Coefficients in the expansion of derivatives in terms of central differences.*

The coefficients in the expansion of the first 52 derivatives in terms of mean central and central differences (in most cases up to the 42nd difference) were computed by the Mathematical Tables Project. For the first 30 derivatives the exact values of the coefficients are given in the form of ordinary fractions for the first 30 differences and for some differences beyond the 30th. For all derivatives beyond the 30th exact fractional values are given for the coefficients of differences up to orders varying between the 41st and 52nd. Finally for most of the coefficients of differences of orders varying from the 31st to the 42nd, 18 significant figures are given. All fractions are believed to be in lowest terms. The tabulated coefficients are valuable in the evaluation of analytic functions of a complex variable when known along a straight line within the region. The coefficients were checked by means of two recursion formulas, which are not mentioned in the literature. (Received April 8, 1943.)

Statistics and Probability

197. Jacob Wolfowitz: *Asymptotic distributions of ascending and descending runs.*

Let $a_1, a_2, \cdots, a_N$ be any permutation of N unequal numbers. Let there be assigned to each permutation the same probability. An element a_i $(1 < i < N)$ is called a turning point if a_i is greater than or less than both a_{i-1} and a_{i+1}. Let a_j and a_{j+k} be consecutive turning points; they are said to determine a "run" of length k. The author

obtains the asymptotic distributions of a large class of functions of these runs. An example of his results is the following: It is proved that the following are asymptotically normally distributed: (a) the total number of runs; (b) $R(p)$, the number of runs of length p; (c) $R(p)$ and $R(q)$ jointly. Similar results are obtained for runs defined by any of a large set of criteria, of which the one given above is of value in statistical applications. (Received May 1, 1943.)

TOPOLOGY

198. Paul Alexandroff: *On homological situation properties of complexes and closed sets.*

The purpose of this paper is to find and to study topological invariants which connect the homological properties of a space K with those of its closed subset A and of the open complement $G=K\setminus A$, and thus contribute to characterize from the homological point of view the *situation of A in K*. Thus the paper constitutes an extension of results already known when K is simply connected (when its β groups are 0 and also when K is a manifold). (Received April 3, 1943.)

NEW PUBLICATIONS

BELL, C., and THOMAS, T. Y. Essentials of plane and spherical trigonometry. New York, Holt, 1943. 6+152 pp. $1.80.

BRENKE, W. C. Plane and spherical trigonometry. New York, Dryden, 1943. 10+269 pp. $1.90.

CAMM, F. J. Mathematical tables and formulae. London, Newnes, 1943. 144 pp. 3s. 6d.

CRAIG, H. V. Vector and tensor analysis. New York and London, McGraw-Hill, 1943. 14+434 pp. $3.50.

HAMMOND, J. R. Concise spherical trigonometry with applications and reviews of solid geometry and plane trigonometry. Boston, Houghton Mifflin, 1943. 13+256 pp. $2.20.

HYSLOP, J. M. Infinite series. Edinburgh and London, Oliver and Boyd; New York, Interscience, 1942. 11+120 pp. $1.75.

MCCREA, W. H. Analytical geometry of three dimensions. Edinburgh and London, Oliver and Boyd; New York, Interscience, 1942. 7+144 pp. $1.75.

MARGENAU, H., and MURPHY, G. M. The mathematics of physics and chemistry. New York, Van Nostrand, 1943. 12+581 pp. $6.50.

Miscellaneous physical tables. Planck's radiation functions and electronic functions. New York, Work Projects Administration, 1941. 7+61 pp. $1.50.

MURPHY, G. M. See MARGENAU, H.

Selected topics in higher mathematics for teachers. Association of Teachers of Mathematics of New York City, 1943. 107 pp. $.50.

THOMAS, T. Y. See BELL, C.

WILKS, S. S. Mathematical statistics. Princeton University Press, 1943. 12+284 pp. $3.75.